Generalized Berezin quantization, Bergman metrics and fuzzy laplacians

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ABSTRACT: We study extended Berezin and Berezin-Toeplitz quantization for compact Kähler manifolds, two related quantization procedures which provide a general framework for approaching the construction of fuzzy compact Kähler geometries. Using this framework, we show that a particular version of generalized Berezin quantization, which we baptize "Berezin-Bergman quantization", reproduces recent proposals for the construction of fuzzy Kähler spaces. We also discuss how fuzzy laplacians can be defined in our general framework and study a few explicit examples. Finally, we use this approach to propose a general explicit definition of fuzzy scalar field theory on compact Kähler manifolds.

Contents

1. Introduction

The quantization of Kähler manifolds seems to play an increasingly important role in certain areas of field and string theory. In particular, recent work on fuzzy geometry, which is partly inspired by string theory, suggests that certain versions of "geometric" quantization provide a framework for a better understanding of fuzzy spaces. In particular, it was proposed in [1] that a specific quantization procedure leads to a general definition of fuzzy compact Kähler manifolds. While originally formulated in terms of an explicit embedding in projective space, this procedure has an intrinsic geometric meaning, which we explore and clarify in the present paper.

It is perhaps not surprising that fuzzy Kähler geometry turns out to be intimately related with Berezin quantization [2] of compact Kähler manifolds, which was studied in [3 – 5] and more recently in [6]–[14] in its Berezin-Toeplitz variant. As we will show, however, the connection involves a few interesting twists. For example, the proposal of [1] is not ordinary (or classical) Berezin or Berezin-Toeplitz quantization in the sense of loc. cit., but rather a modified "Berezin-Bergman" version, which can itself be viewed as a particular realization of a more general Berezin-like procedure. To fully clarify the situation, we introduce *generalized* Berezin and Toeplitz quantizations of compact Kähler manifolds and show how the proposal of [1] fits into this larger framework.

In classical Berezin quantization [3 – 5], one starts with a compact Hodge manifold (X, ω) (where ω is the symplectic form) endowed with a Hermitian holomorphic line bundle L whose Chern connection has curvature equal to $-2\pi i \omega$. Using the Hermitian metric h of L and the volume form of ω, one constructs L^2-scalar products $\langle\ ,\ \rangle_k$ on the spaces of holomorphic sections $E_k := H^0(L^{\otimes k})$ of the positive tensor powers of L. One then performs Berezin quantization at each sufficiently large level k using the coherent states of the finite-dimensional Hilbert spaces $(E_k, \langle\ ,\ \rangle_k)$. The coherent states define Berezin quantization maps $Q_k : \Sigma_k \to \mathrm{End}(E_k)$, where Σ_k are finite-dimensional subspaces of $\mathcal{C}^\infty(X)$. A closely related quantization procedure known as Toeplitz quantization was studied in [6 – 10, 12, 13]. This prescription has better asymptotic properties and is related to Berezin quantization via a geometric version of the Berezin transform. Both quantization prescriptions depend only on the data (X, L, h) – which determines[1] ω; however, we will often use the redundant parameterization (X, ω, L, h) for reasons of notational clarity.

The extension discussed in the present paper starts with the observation that the Berezin quantization maps Q_k at each fixed level k depend only on the holomorphic bundle $L^{\otimes k}$ and on the Hermitian scalar product on its space of holomorphic sections. Hence the entire procedure can be generalized by replacing the L^2-products $\langle\ ,\ \rangle_k$ with an arbitrary sequence of Hermitian scalar products $(\ ,\ )_k$ on the spaces E_k. This results in what we

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[1]Notice that (X, ω, L) determines h up to a constant factor.

call *generalized Berezin quantization* of the Hodge manifold (X, ω). While classical Berezin quantization depends on the data (X, L, h), its generalized version depends on (X, ω, L) and on the sequence of scalar products $(\ , \ )_k$ on the spaces $H^0(L^{\otimes k})$, where L is a holomorphic line bundle such that $c_1(L) = [\omega]$. This gives a large class of apparently novel quantizations of (X, ω). A similar extension exists for Toeplitz quantization and depends on the same data plus the choice of a Radon measure on X. It is related to the corresponding generalized Berezin quantization via an extension of the geometric Berezin transform.

Using this framework, we will show that the procedure proposed in [1] amounts to performing generalized Berezin quantization with respect to a certain series of scalar products on E_k which are induced in an intrinsic manner from a given scalar product on $E_1 = H^0(L)$. This quantization prescription, which we shall call *Berezin-Bergman quantization*, depends only on the data $(X, \omega, L, (\ , \ ))$ and generally differs from the classical Berezin quantization based on (X, ω, L, h). It is intimately related with a certain sequence of Bergman metrics [15] on X and might be of interest in studies of Kähler metrics of constant scalar curvature. Berezin-Bergman quantization has a series of simplifying features which make it eminently computable. In particular, it is rather straightforward to determine the associated quantum objects in this prescription. As an application, we consider a sequence of truncated Laplace operators inspired by this quantization scheme, which can be used to approximate the spectrum of the full Laplacian. These truncated Laplacians correspond to the standard fuzzy Laplacian in the case of complex projective spaces. Explicit numerical computations are presented for the quadratic and cubic Fermat curves in $\mathbb{P}^2$ (the complex planar conic and an elliptic curve, respectively).

We will show that classical Berezin and Berezin-Bergman quantization agree for the case of complex projective spaces, in which they both recover the usual fuzzy geometry construction. The reason for this agreement is due to the fact that $\mathbb{P}^n = \mathrm{U}(n+1)/(\mathrm{U}(n) \times \mathrm{U}(1))$ is a Kähler homogeneous space, and that both quantization prescriptions are compatible with the transitive $\mathrm{U}(n+1)$ action. Since they agree for the only well-studied examples of fuzzy compact Kähler spaces, it follows that both quantization schemes provide potential definitions of general 'fuzzy compact Kähler manifolds'. Which of these one chooses to use depends on the desired asymptotic properties in the classical (i.e. large k) limit, which are currently well understood only for classical Berezin quantization. Thus one could as well choose the latter as a general definition of fuzzy Kähler geometry.

Generalized Berezin-Toeplitz quantization provides a precise framework for "lifting" operators from the space of functions to the quantum Hilbert space, an operation which we call *Berezin-Toeplitz lift*. We propose to define the "fuzzy" Laplacian of a compact Hodge manifold as the Berezin-Toeplitz lift of the Laplace operator through the Berezin-Toeplitz quantization of that manifold. Together with the integral representation of the trace also derived in the present paper, the Berezin-Toeplitz lift of the Laplacian enables us to give an explicit definition of fuzzy scalar field theory on arbitrary compact Hodge manifolds. This construction might be of interest for the fuzzy field theory community.

The paper is organized as follows. In section 2, we recall some facts about polarizations and quantum line bundles, mostly in order to fix our notations and terminology. We also

discuss Bergman metrics and metrized Kodaira embeddings and introduce a "relative" version of Rawnsley's epsilon function [3], which will prove useful for our purpose. In section 3, we introduce the generalized Berezin and Berezin-Toeplitz quantization defined by a sequence of scalar products and explore their basic properties. In particular, we discuss the relation between the two extended quantization procedures and address the effect of changing the scalar products. We also discuss the notion of relatively balanced Bergman metrics, which enters naturally in our set-up, as well as the lift of linear operators from the space of functions to the quantization space. Finally, we construct the generalized Berezin (coherent state) product and give the description of the quantization in the language of star algebras. In section 4, we recall the basic properties of classical Berezin and Berezin-Toeplitz quantization and in particular their asymptotic behavior, which allows for the construction of the associated formal deformation quantizations [7, 8, 10, 12]. We also briefly discuss the classical quantization of affine and projective spaces, which will be used later. For projective spaces, we follow an approach which recovers the formalism used in [1], showing that the notion of fuzzy projective spaces [17] coincides with the classical Berezin quantization of those spaces. section 5 gives the general description of Berezin-Bergman quantization. After presenting the intrinsic formulation, we show how one can recover the description through embeddings in $\mathbb{P}^n$ and clarify some of its basic properties. We also show that the Berezin-Bergman quantization of $\mathbb{P}^n$ coincides with the classical Berezin quantization of the latter, an accident[2] which is due to the fact that complex projective spaces are homogeneous Kähler manifolds. Section 6 takes up the issue of "quantized harmonic analysis" in the framework of classical Berezin quantization. We discuss two ways of constructing a fuzzy Laplace operator. One is used to compute the approximate spectrum of harmonic functions on Fermat curves, while the other one appears in the construction of fuzzy scalar field theory on arbitrary compact Hodge manifolds.

2. Polarizations, quantum line bundles and Bergman metrics

2.1 Polarizations and quantum line bundles

Consider a connected compact complex manifold X of complex dimension n. Recall that a *polarization* of X is a positive holomorphic line bundle L over X. Given a polarized complex manifold (X, L), there exists a positive integer k_0 such that the tensor powers $L^k := L^{\otimes k}$ are very ample for all $k \geq k_0$; in particular, X can be presented as a projective algebraic variety by the Kodaira embedding determined by L^k for any $k \geq k_0$.

A Kähler form ω on X is called *integral* if its cohomology class $[\omega]$ belongs to $H^2(X, \mathbb{Z})$; in this case, (X, ω) is called a *Hodge manifold*. Given a polarization L of X, the Kähler form is called *L-polarized* if $[\omega]$ equals $c_1(L)$. In this case, L is called a *Kähler polarization* of (X, ω) and the triple (X, L, ω) is called a *polarized Hodge manifold*. It is well-known that any Hodge manifold (X, ω) admits Kähler polarizations; moreover, the isomorphism classes of Kähler polarizations for (X, ω) form a torsor under the Abelian group $\text{Pic}^0(X)$

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[2]Such a simple relation between Berezin-Bergman and Berezin quantization should not be expected for general compact Hodge manifolds.

of degree zero holomorphic line bundles. In particular, Kähler polarizations of (X, ω) are unique up to isomorphism when X is simply connected. Conversely, a polarized complex manifold (X, L) admits Kähler metrics whose Kähler class equals $c_1(L)$.

Given a polarized manifold (X, L), there exists a well-known bijection between L-polarized Kähler metrics on X and homothety (positive constant prefactor rescaling) classes of Hermitian bundle metrics on L. Recall that this bijection is determined as follows:

(a) Given a Hermitian metric h on L, there exists a unique Kähler metric on X whose Kähler form satisfies $\omega = \frac{i}{2\pi} F$, where F is the curvature of the Chern connection[3] ∇ of (L, h). This Kähler metric is automatically L-polarized since $c_1(L) = \frac{i}{2\pi}[F]$. Multiplying h by a positive constant does not change the associated Kähler metric.

(b) Given an L-polarized Kähler form ω, there exists a Hermitian metric h on L such that $\omega = \frac{i}{2\pi} F$, where F is the curvature of the Chern connection of (L, h). The metric h is determined by ω up to multiplication by a positive constant. Such a Hermitian line bundle (L, h) is sometimes called a *quantum line bundle* for (X, ω), and one says that (X, ω, L, h) is a *prequantized Hodge manifold*. Two quantum line bundles (L, h) and (L', h') for (X, ω) are called *equivalent* if there exists an isomorphism $\psi : L \to L'$ of holomorphic line bundles such that $\psi^*(h') = h$. Equivalence classes of quantum line bundles for (X, ω) form a $\mathrm{Hom}(\pi_1(X), S^1)$-torsor.

Given a quantum line bundle (L, h), one can endow L^k with the induced metric $h_k = h^{\otimes k}$ and with the corresponding Chern connection $\nabla_k = \nabla^{\otimes k}$. Then $\omega = \frac{i}{2\pi k} F_k$, where $F_k = kF$ is the curvature of ∇_k. A fixed positive measure μ on X induces a Hermitian scalar product on the space of smooth sections $\Gamma(L^k)$:

$$\langle s_1, s_2 \rangle_k^{\mu,h} := \int_X d\mu \ h_k(s_1, s_2) \ . \tag{2.1}$$

We let $L_k^2(L, h, \mu)$ be the L^2-completion of $\Gamma(L^k)$ with respect to this scalar product. The finite-dimensional subspace $H^0(L^k) \subset \Gamma(L^k)$ of holomorphic sections inherits a scalar product, which we denote by the same symbol. The standard choice for μ is the Liouville measure determined by the canonical volume form $\frac{\omega^n}{n!}$ of (X, ω):

$$\langle s_1, s_2 \rangle_k^h := \int_X \frac{\omega^n}{n!} \ h_k(s_1, s_2) \ , \tag{2.2}$$

however, it is often useful to work more generally. For example, one has another natural measure — namely that defined by the volume form $\Omega \wedge \bar{\Omega}$ — when (X, ω) is algebraically a Calabi-Yau manifold, (i.e. when $c_1(TX) = 0$) with holomorphic top form Ω. In these cases, one is often interested in Kähler forms ω in a given integral cohomology class, which however differ from the Kähler form ω_{CY} of the Calabi-Yau metric in that class. (Thus one has $[\omega] = [\omega_{\mathrm{CY}}] = c$ for some positive class $c \in H^2(X, \mathbb{Z})$ but $\omega \neq \omega_{\mathrm{CY}}$.) Recall that ω_{CY} is not explicitly known in practice. In such a situation, one has $\frac{\omega^n}{n!} \neq \Omega \wedge \bar{\Omega} = \frac{\omega_{\mathrm{CY}}^n}{n!}$.

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[3]The Chern connection is the unique connection on (L, h) which is both Hermitian and compatible with the holomorphic structure.

An *automorphism* of a prequantized Hodge manifold (X, ω, L, h) is a pair $\gamma := (\gamma_0, \gamma_1)$ such that γ_0 is a holomorphic isometry of (X, ω) and γ is a holomorphic bundle isometry of (L, h) above γ_0. In particular, $\gamma_1(x)$ is an isometry from $(L_x, h(x))$ to $(L_{\gamma_0(x)}, h(\gamma_0(x)))$ for all $x \in X$. The automorphisms of (X, ω, L, h) form a group which we denote by $\mathrm{Aut}(X, \omega, L, h)$. This group acts linearly on the space of sections $H^0(L^k)$ via:

$$\rho_k(\gamma)(s) = \gamma_1^{\otimes k} \circ s \circ \gamma_0^{-1} \quad (s \in H^0(L^k)) \quad . \tag{2.3}$$

The actions $\rho_k : \mathrm{Aut}(X, \omega, L, h) \rightarrow \mathrm{End}(H^0(L^k))$ are unitary with respect to the L^2-scalar product (2.1) provided that the measure μ is invariant under the group $\mathrm{Aut}(X, \omega)$ of holomorphic isometries of ω. This is the case, for example, when μ is the Liouville measure defined by ω.

An automorphism γ is called *trivial* if $\gamma_0 = \mathrm{id}_X$ and γ_1 is given by $\gamma_1(x) = (e^{i\alpha})\cdot$ for all x, where α is a real constant. Thus $\mathrm{Aut}(X, \omega, L, h)$ always contains a $\mathrm{U}(1)$ subgroup. The quotient $\mathrm{Aut}(X, \omega, L, h)/\mathrm{U}(1)$ is the subgroup $\mathrm{Aut}_{L,h}(X, \omega) \subset \mathrm{Aut}(X, \omega)$ of those holomorphic isometries γ_0 of (X, ω) which admit a lift $\gamma_1 : L \rightarrow L$ such that (γ_0, γ_1) is an automorphism of (X, ω, L, h). Thus we have an exact sequence of groups [3]:

$$1 \rightarrow \mathrm{U}(1) \rightarrow \mathrm{Aut}(X, \omega, L, h) \rightarrow \mathrm{Aut}_{L,h}(X, \omega) \rightarrow 1 \quad . \tag{2.4}$$

In general, the inclusion $\mathrm{Aut}_{L,h}(X, \omega) \subset \mathrm{Aut}(X, \omega)$ is strict, i.e. not every holomorphic isometry admits a lift. The obstruction to the existence of such a lift lives in the group $\mathrm{Hom}(\pi_1(X), S^1)$, so in particular $\mathrm{Aut}_{L,h}(X, \omega)$ equals $\mathrm{Aut}(X, \omega)$ when X is simply connected. Notice that $\mathrm{Aut}(X, \omega)$ is usually discrete since a generic Hodge manifold has no continuous holomorphic isometries. The case usually studied in the fuzzy literature (namely that of rather special homogeneous spaces) is highly non-generic in this regard.

Remark. A holomorphic section σ of L which is not identically zero yields a local frame above the open set $U_\sigma := \{x \in X | \sigma(x) \neq 0\}$. With respect to this frame, the Chern connection ∇ of (L, h) is given by $\nabla \equiv d + \partial \log h(\sigma, \sigma)$, where $d = \partial + \bar{\partial}$ and ∂ are the de Rham and Dolbeault operators. Its curvature is $F = -\partial\bar{\partial} \log h(\sigma, \sigma) = -2\pi i\omega$. Hence the function $K_\sigma := -\log h(\sigma, \sigma)$ defines a local Kähler potential on U_σ:

$$\omega = \frac{i}{2\pi} \partial\bar{\partial} K_\sigma \quad .$$

Every section $s \in \Gamma(L^k)$ can be written above U_σ in the form $s = f\sigma^{\otimes k}$, where f is a smooth complex-valued function on U_σ, which is holomorphic iff s is holomorphic. When the measure μ satisfies $\mu(X \setminus U_\sigma) = 0$, this gives isometries of $\Gamma(L^k)$ and $H^0(L^k)$ with the spaces of smooth, respectively holomorphic functions on U_σ endowed with the scalar product:

$$\langle f, g \rangle_{k,\sigma} = \int_{U_\sigma} d\mu \, e^{-kK_\sigma} \bar{f}g \quad . \tag{2.5}$$

It follows that $L_k^2(L, h, \mu)$ can be identified with the space $L^2(U_\sigma, e^{-kK_\sigma}\mu)$.

2.2 Parameterizing hermitian bundle metrics and polarized Kähler forms

Fixing a polarized complex manifold (X, L), let $\mathbb{L}$ be the total space of L and $\mathbb{L}_0$ be the total space with the graph o of the zero section removed. Hermitian metrics h on L are uniquely determined by their square norm functions $\hat{h} \in \mathcal{C}^\infty(\mathbb{L}_0, \mathbb{R}_+)$:

$$\hat{h}(q) := h(q, q) , \quad q \in \mathbb{L} .$$

These are smooth non-negative functions on $\mathbb{L}$, strictly positive on $\mathbb{L}_0$ and having the property $\hat{h}(cq) = |c|^2 \hat{h}(q)$ for all $q \in \mathbb{L}$ and all $c \in \mathbb{C}$ (this property implies $\hat{h}|_o = 0$). The set $\mathrm{Met}(L)$ of Hermitian metrics on L can be identified with the set of all such functions on $\mathbb{L}$ and thus forms an infinite-dimensional convex cone in $\mathcal{C}^\infty(\mathbb{L}, \mathbb{R})$. As a consequence, L-polarized Kähler metrics are parameterized by rays in this real cone. If we fix a reference metric h_0 on L, then any other metric h is described by the smooth positive function $\phi = \frac{\hat{h}}{\hat{h}_0}$ on X, and we find that $\mathrm{Met}(L)$ can also be identified with $\mathcal{C}^\infty(X, \mathbb{R}_+^*)$. Taking the logarithm $\psi = \log \phi$, this gives bijections between $\mathrm{Met}(L)$ and $\mathcal{C}^\infty(X, \mathbb{R})$, as well as between the set of L-polarized Kähler metrics and the space $\{\psi \in \mathcal{C}^\infty(X, \mathbb{R}) | \psi(x) = 0\}$, where x is any fixed point of X.

In this paper, we will use a slightly different parameterization in the case when L is very ample. For any $q \in \mathbb{L}_0$, we let $\hat{q} : H^0(L) \to \mathbb{C}$ be the linear functional (called *evaluation functional*) defined through:

$$s(\pi(q)) = \hat{q}(s)q, \quad s \in H^0(L), \tag{2.6}$$

where $\pi : \mathbb{L} \to X$ is the bundle projection. We have the obvious property $\widehat{cq} = \frac{1}{c}\hat{q}$ for all non-vanishing complex numbers c. The very ampleness of L implies $\hat{q} \neq 0$ for all $q \in \mathbb{L}_0$.

A Hermitian scalar product $(\,,\,)$ on the finite-dimensional space $H^0(L)$ induces a scalar product on the dual space $H^0(L)^* = \mathrm{Hom}_\mathbb{C}(H^0(L), \mathbb{C})$, which allows us to consider the Hermitian metric h_B on L whose square norm function is given by:

$$\hat{h}_B(q) = \frac{1}{||\hat{q}||^2} \quad (q \in \mathbb{L}_0) \tag{2.7}$$

(and $\hat{h}_B|_o = 0$). This is called the *Bergman metric*[4] [15] on L defined by the scalar product $(\,,\,)$. Since we now have a reference Hermitian metric on L, we can describe any other metric h via the positive function:

$$\epsilon := \frac{\hat{h}}{\hat{h}_B} \in \mathcal{C}^\infty(X, \mathbb{R}_+^*) , \tag{2.8}$$

which we call the *epsilon function of h relative to* $(\,,\,)$:

$$h(q, q) = \epsilon(\pi(q)) h_B(q, q) . \tag{2.9}$$

Thus Hermitian metrics on L are parameterized by their relative epsilon functions, once we fixed a scalar product on $H^0(L)$.

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[4]The name of these metrics honors the work of the mathematician Stefan Bergman.

The relative epsilon function defined above depends on h and on the scalar product chosen on $H^0(L)$, and is a generalization of the more familiar object considered in [3–5]. To make contact with the latter, notice that fixing h gives a distinguished choice of a scalar product on $H^0(L)$, namely the L^2-product $\langle\ ,\ \rangle$ defined by h and by the Liouville measure of the associated Kähler form ω. The epsilon function of h with respect to this L^2-scalar product depends on h only (remember that ω is determined by h), and will be called the *absolute epsilon function* of h. The latter is the epsilon function considered in [3–5].

The L-polarized Kähler metric on X determined by h_B is called the *Bergman metric on X induced by* $(\ ,\ )$. Its Kähler form is denoted by ω_B. The Kähler form ω determined by the Hermitian bundle metric (2.9) takes the form:

$$\omega = \omega_B - \frac{i}{2\pi}\partial\bar{\partial}\log\epsilon\ ,$$

so as expected we have $\omega = \omega_B$ iff the relative epsilon function of h is constant. Since ω determines h up to multiplication by a constant, it also determines the relative epsilon function of the latter up to the same ambiguity. We will see below that the L-polarized Bergman metrics are those metrics induced on X by pulling-back Fubini-Study metrics through the Kodaira embedding $i : X \hookrightarrow \mathbb{P}[H^0(L)^*]$ determined by the very ample line bundle L, where the Fubini-Study metric being pulled-back is determined by the scalar product on $H^0(L)^*$.

Remarks.

1. Let $n + 1 := \dim_{\mathbb{C}} H^0(L)$ and pick an arbitrary basis $s_0 \ldots s_n$ of $H^0(L)$. Setting $G_{ij} := (s_i, s_j)$, we have:

$$||\hat{q}||^2 = \sum_{i,j=0}^{n} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)\ \ (q \in \mathbb{L}_0)\ ,$$

where G^{ij} are the entries of the inverse matrix to (G_{ij}):

$$\sum_{j=0}^{n} G^{ij}G_{jk} = \delta_{ik}\ .$$

The norm square with respect to the bundle Bergman metric determined by $(\ ,\ )$ takes the form:

$$h_B(q, q) = \frac{1}{\sum_{i,j=0}^{n} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)}\ \ (q \in \mathbb{L}_0)\ ,$$

while the epsilon function relative to $(\ ,\ )$ of an arbitrary Hermitian metric h on L is given by:

$$\epsilon(x) = \sum_{i,j=0}^{n} G^{ij}h(x)(s_i(x), s_j(x))\ .$$

The Hermitian metric h is given as follows in terms of its relative epsilon function:

$$h(q, q) = \epsilon(x)h_B(q, q) = \frac{\epsilon(x)}{\sum_{i,j=0}^{n} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)}\ .$$

2. Bergman bundle metrics on L are in bijection with Hermitian products on $H^0(L)$, which form the non-compact homogeneous space $U(n+1,\mathbb{C}) \setminus GL(n+1,\mathbb{C})$ under the action of $GL(n+1,\mathbb{C}) \simeq GL(H^0(L))$. They are extremely special in the set of all Hermitian bundle metrics on L. Correspondingly, L-polarized Bergman metrics on X are extremely special among L-polarized Kähler metrics.

3. The L^2-scalar product on $H^0(L)$ defined by h_B and by the volume form of ω_B:

$$\langle s,t \rangle = \int_X \frac{\omega_B^n}{n!} h_B(s,t) \quad (s,t \in H^0(L))$$

need not coincide with the scalar product $(\ ,\ )$ which parameterizes h_B. If they do, one says that the scalar product $(\ ,\ )$ and associated Bergman bundle and manifold metrics h_B, ω_B are *balanced* [18]. It is clear that ω_B is balanced iff its *absolute* epsilon function is constant; Hermitian line bundles (L, h_B) endowed with balanced bundle metrics were called *regular* in [4, 5]. It was shown in [18] that a balanced scalar product on $H^0(L)$ is unique up to a constant scale factor if it exists, so L-polarized balanced metrics on X are at most unique. A polarized complex manifold (X, L) is called balanced if $H^0(L)$ admits a balanced scalar product. When L is very ample, it is known (see e.g. [19, 20]) that (X, L) is balanced iff its Kodaira embedding $i(X)$ is Chow-Mumford stable in the projective space $\mathbb{P}[H^0(L)^*]$.

2.3 Bergman metrics from metrized Kodaira embeddings

Let X be a compact complex manifold. By the Kodaira embedding theorem, a very ample line bundle L gives a holomorphic embedding $i : X \hookrightarrow \mathbb{P}V$, where $V = E^*$ and $E := H^0(L)$ is the space of holomorphic sections of L, whose complex dimension we denote by $n+1$. The embedding allows us to view X as a regular projective variety in $\mathbb{P}V$, whose homogeneous coordinate ring $R(X, L) = \oplus_{k \geq 0} H^0(L^k)$ is generated in degree one. In particular, L and the pull-back $i^*(H)$ of the hyperplane bundle $H := \mathcal{O}_{\mathbb{P}V}(1)$ are isomorphic as holomorphic line bundles.

Conversely, if we are given any smooth projective variety X in a projective space $\mathbb{P}V$ whose vanishing ideal $I(X)$ is generated in degrees greater than one, then the restriction $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}V}(1)|_X$ is very ample and the embedding $X \hookrightarrow \mathbb{P}V$ can be viewed as the Kodaira embedding determined by this restriction. The space of holomorphic sections of $\mathcal{O}_X(1)$ identifies with the vector space $E = V^*$.

A *metrized Kodaira embedding* is a Kodaira embedding determined by a very ample line bundle L on X together a fixed choice of a Hermitian scalar product $(\ ,\ )$ on its space of holomorphic sections $E := H^0(L)$. For such embeddings, the scalar product on E induces a scalar product on $V = E^*$, which makes $\mathbb{P}V$ into a (finite-dimensional) projective Hilbert space. The latter carries the Fubini-Study metric[5] determined by the scalar product. Its Kähler form is given by:

$$\pi^*(\omega_{\mathrm{FS}})(v) = \frac{i}{2\pi} \partial \bar{\partial} \log(||v||^2) \ ,$$

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[5]Recall that homogeneous Kähler metrics on $\mathbb{P}V$ are in bijection with Hermitian scalar products on E taken up to constant rescaling, and these are the Fubini-Study metrics. They are all related by $PGL(E)$-transformations.

where $\pi : V \to \mathbb{P}V$ is the canonical projection while $|| \ ||$ is the norm induced on $V = E^*$. There exists a one to one correspondence between metrized Kodaira embeddings of X and holomorphic embeddings in finite-dimensional projective Hilbert spaces such that the vanishing ideal of the embedding is generated in degrees greater than one.

The Fubini-Study metric admits the hyperplane bundle H as a quantum line bundle, when the latter is endowed with the Hermitian bundle metric h_{FS} induced from E. Since $L \simeq i^*(H)$ as holomorphic line bundles, the pull-back $i^*(h_{\text{FS}})$ defines a Hermitian metric h_B on L. The latter coincides with the Bergman bundle metric determined by (,). The pulled-back Kähler form $\omega_B = i^*(\omega_{\text{FS}})$ admits (L, h_B) as a quantum line bundle, and coincides with the Bergman Kähler form determined by (,). It follows that Bergman metrics on X coincide with pull-backs of Fubini-Study metrics via metrized Kodaira embeddings.

Remark. A choice of basis $z_0 \dots z_n$ for $E = V^*$ allows us to express $v \in V$ as: $v = \sum_{i=0}^n v_i e_i$, where (e_i) is the basis of V dual to (z_i) and $v_i = z_i(v)$. This gives an identification of V with the space $\mathbb{C}^{n+1}$ endowed with the scalar product given by $\langle u, v \rangle = \sum_{i,j=0}^n G^{ij} \bar{u}_i v_j$, where the G^{ij} are given as above. Then $\mathbb{P}V$ identifies with $\mathbb{P}^n$ endowed with the Fubini-Study metric defined by this scalar product. It is customary to choose an orthonormal basis, in which case the Fubini-Study metric takes the familiar form in homogeneous coordinates. In this case, the freedom of choosing the scalar product (,) is replaced by the freedom of acting with $PGL(n+1, \mathbb{C})$ transformations on the homogeneous coordinates of $\mathbb{P}^n$.

3. Generalized Berezin and Toeplitz quantization

Two related general methods for quantizing compact Hodge manifolds are provided by the so-called Berezin and Berezin-Toeplitz quantization, which were studied in [3–5] and [6, 8–10, 12–14]. This quantization scheme realizes ideas going back to [2] in a modified and extended form. In this approach, one starts with a prequantized Hodge manifold (X, ω, L, h) and considers the sequence of Hermitian vector spaces $(E_k := H^0(L^k), \langle \ , \ \rangle_k)$ for $k \geq k_0$, where k_0 is a positive integer k such that L^k is very ample for all $k \geq k_0$. The Hermitian scalar products $\langle \ , \ \rangle_k$ are taken to be the L^2-products (2.2) induced by h and by the Liouville measure of ω. At every level k, the Hermitian structure makes $H^0(L^k)$ into a reproducing kernel Hilbert space, and in particular allows one to introduce coherent vectors, which are special holomorphic sections of L^k parameterized by the points of X. Using these vectors, one defines Berezin symbol maps $\sigma_k : \text{End}(E_k) \to \mathcal{C}^\infty(X)$, which turn out to be injective due to compactness of X. The inverses on the images Σ_k of these maps provide bijections $Q_k : \Sigma_k \to \text{End}(E_k)$ which are known as *Berezin quantization maps*. The collection $(Q_k)_{k \geq k_0}$ of such maps constitutes the *classical Berezin quantization* of (X, ω) induced by the quantum line bundle (L, h).

A fundamental problem in this approach is to describe the asymptotic behavior of Q_k for large k. One relevant problem is whether the sequence Q_k defines in some manner a formal deformation quantization of X, and to identify the corresponding formal star product. It turns out that these questions can be answered quite elegantly by considering a variation of Berezin's approach, which is known as classical Berezin-Toeplitz or

simply *classical Toeplitz quantization*. This modified quantization prescription consists of replacing Q_k by the so-called Toeplitz quantization maps $T_k : \mathcal{C}^\infty(X) \to \mathrm{End}(E_k)$, which are constructed as integral operators with the help of the coherent state projector. The asymptotic behavior of $T_k(f)$ can be controlled using results of de Monvel, Guillemin and Sjöstrand [23, 24], allowing one to prove [7, 8, 10, 12] that Toeplitz quantization gives rise to a formal star product and thus to a formal deformation quantization of (X, ω). Since Berezin and Toeplitz quantization turn out to be related via a general version of the Berezin transform (which corresponds to a "change of operator ordering"), this also allows one to construct a formal star product corresponding to Berezin quantization [12] (see [14] for a different approach).

The construction of classical Berezin and Toeplitz quantizations can be generalized by considering an arbitrary sequence of scalar products $(\ ,\ )_k$ on the spaces $H^0(L^k)$ instead of the L^2-products (2.2). This leads to what we call *generalized Berezin and Toeplitz quantizations*. In this section, we discuss the basic properties of the resulting quantization schemes. As in the classical case, an important question — which we do not attempt to settle here — concerns the asymptotic behavior of these generalized quantizations for large k, which will of course depend markedly on the choice of scalar products.

When studying the situation at each fixed level $k \geq k_0$, the replacement $L \to L^k$ allows us to work with a very ample line bundle L while dropping the index k from the notation. Let us therefore fix a compact complex manifold X, a very ample line bundle L on X and a Hermitian scalar product $(\ ,\ )$ on the vector space $E := H^0(L)$, whose dimension we denote by $N + 1$. In the following we will sometimes consider an arbitrary basis $s_0 \dots s_N$ of $H^0(L)$. In this case, we let G be the Hermitian positive matrix with entries $G_{ij} := (s_i, s_j)$ and G^{ij} be the entries of the inverse matrix G^{-1}. Any section $s \in E$ can be expanded as:

$$s = \sum_{i,j=0}^{N} G^{ij}(s_j, s)s_i \ \ .$$

3.1 Coherent states

Given $q \in \mathbb{L}_0$, consider the evaluation functional $\hat{q}$ on E defined in (2.6). By Riesz's theorem, there exists a unique holomorphic section $e_q \in E$ such that $(e_q, s) = \hat{q}(s)$ for all $s \in E$. Direct computation gives the explicit expression:

$$e_q = \sum_{i,j=0}^{N} G^{ji}\overline{\hat{q}(s_i)}s_j \ \ ,$$

which implies:

$$||e_q||^2 = \sum_{i,j=0}^{N} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j) \ \ .$$

Notice that e_q cannot be the zero section, since that would imply that all sections of L vanish at $x = \pi(q)$, which is impossible since L is very ample. The element e_q of E is called the *Rawnsley coherent vector* [3] defined by q. Also notice that e_q depends only on the scalar product chosen on E.

If q' is another non-vanishing element of the fiber L_x, then $q' = cq$ for some non-vanishing complex number c and we have $e_{q'} = \frac{1}{c} e_q$. It follows that the complex line $l_x := \langle e_q \rangle = \mathbb{C} e_q \subset E$ depends only on the point $x \in X$. This can be interpreted as follows. Let $\bar{L}$ be the line bundle obtained by reversing the complex structure of all fibers;[6] this is a holomorphic line bundle over the complex manifold $\bar{X}$ obtained by reversing the complex structure of X. The scaling property of coherent vectors implies that the element $e_x := \bar{q} \otimes e_q \in \bar{L}_x \otimes H^0(L)$ depends only on the point $x \in X$. The scalar product on $H^0(L)$ extends to a sesquilinear map taking $[\bar{L}_x \otimes E] \times [\bar{L}_y \otimes E]$ into $\bar{L}_x \otimes \bar{L}_y$. In particular, the combination $K(x, y) = (e_x, e_y)$ defines a holomorphic section K of the external tensor product $\bar{L} \boxtimes \bar{L}$ (which is a holomorphic line bundle over $X \times \bar{X}$). This is the *reproducing kernel* of the finite-dimensional Hilbert space $(H^0(L), (\ ,\ ))$. Also notice that the vector e_q gives a well-defined element $[e_x]$ of the projective space $\mathbb{P}E$, which depends antiholomorphically on $x \in X$. This is called the *Rawnsley coherent state* at x. Thus we have an antiholomorphic embedding $j : X \to \mathbb{P}E$, called the *coherent state embedding* (cf. [16]); it can be viewed as dual to the metrized Kodaira embedding.

Rawnsley's *coherent projectors* are the orthoprojectors on the lines $l_x \subset E$:

$$P_x := \frac{|e_q)(e_q|}{(e_q|e_q)} \quad (q \in L_x \setminus \{0\}) \ . \tag{3.1}$$

They depend only on L, on the point $x \in X$ and on the scalar product chosen on E. Given a linear operator $C \in \mathrm{End}(E)$, its *lower Berezin symbol* is the function $\sigma(C) : X \to \mathbb{C}$ given by:

$$\sigma(C)(x) := \mathrm{tr}(C P_x) = \frac{(e_q|C|e_q)}{(e_q|e_q)} \quad (q \in L_x \setminus \{0\}) \ . \tag{3.2}$$

This gives a linear map $\sigma : \mathrm{End}(E) \to \mathcal{C}^\infty(X)$, whose image we denote by Σ. Notice that σ and Σ depend only on L and on the scalar product $(\ ,\ )$ chosen on E. The obvious relation:

$$\sigma(C^\dagger) = \overline{\sigma(C)}$$

implies that Σ is closed under complex conjugation, i.e. $\bar{\Sigma} = \Sigma$. Also notice that Σ contains the constant unit function $1_X = \sigma(\mathrm{id}_E)$.

3.2 Generalized Berezin quantization

It was shown in [5] that the Berezin symbol map $\sigma : \mathrm{End}(E) \to \mathcal{C}^\infty(X)$ is injective when $(\ ,\ )$ is the L^2-scalar product defined by a Hermitian metric on L and by the Liouville measure of the associated Kähler form. We show below that the Berezin symbol changes as in (3.19) when changing the scalar product. This implies that σ is in fact injective for an arbitrary scalar product on E. Hence the corestriction $\sigma|^\Sigma : \mathrm{End}(E) \to \Sigma$ is a

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[6]Thus the fiber $\bar{L}_x$ coincides with L_x as an additive group, but is endowed with the external multiplication with scalars given by $\alpha * u = \bar{\alpha} u$ for all $\alpha \in \mathbb{C}$ and all $u \in L_x$. The identity map becomes an antilinear involution when viewed as a map from L_x to $\bar{L}_x$. This gives an involution between L and $\bar{L}$, which we denote by an overline.

linear isomorphism and we can associate an operator on E to every function $f \in \Sigma$ via the *generalized Berezin quantization map* $Q = (\sigma|^{\Sigma})^{-1} : \Sigma \to \mathrm{End}(E)$:

$$Q(f) := \sigma^{-1}(f) \quad \forall f \in \Sigma \ . \tag{3.3}$$

The extension from the case of $[3-5]$ is simply that we allow for an arbitrary scalar product on E. The Berezin quantization map depends only on L and on the choice of this scalar product. It satisfies the relations:

$$Q(\bar{f}) = Q(f)^{\dagger} \ , \quad Q(1_X) = \mathrm{id}_E \ .$$

The Berezin star algebra. The *Berezin product* $\diamond : \Sigma \times \Sigma \to \Sigma$ is defined via the formula:

$$f \diamond g := \sigma(Q(f)Q(g)) \Leftrightarrow Q(f \diamond g) = Q(f)Q(g) \ . \tag{3.4}$$

Together with the usual complex conjugation of functions $f \to \bar{f}$, it makes Σ into a unital finite-dimensional associative $*$-algebra. The Berezin quantization map gives an isomorphism of $*$-algebras:

$$Q : (\Sigma, \diamond, \bar{\ }) \to (\mathrm{End}(E), \circ, \dagger) \ .$$

Recall that $(\mathrm{End}(E), \circ, \dagger, || \ ||_{\mathrm{HS}})$ is a B^*-algebra[7] with non-degenerate trace given by the usual trace of operators. It follows that the induced linear map (called the *Berezin trace*):

$$\int f := \mathrm{tr}\, Q(f) \ \ (f \in \Sigma) \tag{3.5}$$

is a nondegenerate trace on the Berezin star algebra $(\Sigma, \diamond, \bar{\ })$:

$$\int \bar{f} = \overline{\int f}$$

$$\int f \diamond g = \int g \diamond f$$

$$\int f \diamond g = 0 \ , \qquad \forall g \in \Sigma \Rightarrow f = 0 \ .$$

Moreover, the scalar product on Σ (called the *Berezin scalar product*) obtained by transporting the Hilbert-Schmidt product:

$$\prec f, g \succ_B := \langle Q(f), Q(g) \rangle_{\mathrm{HS}} = \mathrm{tr}\left(Q(f)^{\dagger} Q(g) \right) \tag{3.6}$$

coincides with the scalar product induced by the Berezin trace:

$$\prec f, g \succ_B = \int \bar{f} \diamond g \ .$$

Since $(\mathrm{End}(E), \circ, \dagger, || \ ||_{\mathrm{HS}})$ is a B^*-algebra, the norm defined by the Berezin product satisfies:

$$||f \diamond g||_B \leq ||f||_B \, ||g||_B$$

$$||\bar{f}||_B = ||f||_B \ .$$

Thus $(\Sigma, \diamond, \bar{\ }, || \ ||_B)$ is a B^*-algebra with non-degenerate trace, and (Q, σ) are mutually inverse isomorphisms of B^*-algebras with trace. Notice that $||1_X||_B = ||\mathrm{id}_E||_{\mathrm{HS}} = N + 1$.

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[7]A B^* algebra is a Banach ($||xy|| \leq ||x|| \, ||y||$) $*$-algebra in which the identity $||x^*|| = ||x||$ is satisfied.

The push and pull of linear operators. The isomorphism Q allows us to transport $\mathbb{C}$-linear operators between Σ and $\text{End}(E)$. Given a linear operator $\mathcal{O} : \Sigma \to \Sigma$, define its *Berezin push* $\mathcal{O}^B : \text{End}(E) \to \text{End}(E)$ via:

$$\mathcal{O}^B := Q \circ \mathcal{O} \circ \sigma \quad \Leftrightarrow Q \circ \mathcal{O} = \mathcal{O}^B \circ Q \quad . \tag{3.7}$$

Given a linear operator $\mathcal{V} : \text{End}(E) \to \text{End}(E)$, define its *Berezin pull* though:

$$\mathcal{V}_B := \sigma \circ \mathcal{V} \circ Q \quad \Leftrightarrow \sigma \circ \mathcal{V} = \mathcal{V}_B \circ \sigma \quad . \tag{3.8}$$

The operations of Berezin push and pull are mutually inverse linear isomorphisms between $\text{End}_{\mathbb{C}}(\Sigma)$ and $\text{End}_{\mathbb{C}}(\text{End}(E))$. They are well-behaved with respect to the Berezin scalar product on Σ in the sense that the following identities hold:

$$\begin{aligned} \prec f, \mathcal{O}(g) \succ_B &= \langle Q(f), \mathcal{O}^B(Q(g)) \rangle_{\text{HS}} \\ \langle C_1, \mathcal{V}(C_2) \rangle_{\text{HS}} &= \prec \sigma(f), \mathcal{V}_B(\sigma(g)) \succ_B \end{aligned} \quad . \tag{3.9}$$

In particular, the Berezin push of a $\prec, \succ_B$-Hermitian operator is $\langle, \rangle_{\text{HS}}$-Hermitian and the Berezin pull of a $\langle, \rangle_{\text{HS}}$-Hermitian operator is $\prec, \succ_B$-Hermitian.

The squared two point function. For later reference, define the squared two-point function $\Psi \in C^\infty(X \times X, \mathbb{R})$ of coherent states:

$$\Psi(x, y) := \text{tr}(P_x P_y) = \sigma(P_y)(x) = \sigma(P_x)(y) = \frac{|(e_x|e_y)|^2}{||e_x||^2 ||e_y||^2} \geq 0 \quad . \tag{3.10}$$

This function is symmetric and non-negative on $X \times X$:

$$\Psi(x, y) = \Psi(y, x) \quad \forall x, y \in X$$

and vanishes at points (x, y) such that e_x is orthogonal to e_y. The vanishing divisor of Ψ is known as the *polar divisor* [16].

Behavior under automorphisms. Recall that the group $\text{Aut}(X, \omega, L, h)$ acts linearly on E (see eq. (2.3)). It is easy to check the relation:

$$\rho(\gamma^{-1})^\dagger(e_q) = e_{\gamma(q)} \quad .$$

Let us assume that the action ρ is $(,)$-unitary:

$$\rho(\gamma)^\dagger = \rho(\gamma)^{-1} \quad .$$

Then the relation above becomes:

$$\rho(\gamma)(e_q) = e_{\gamma(q)}$$

and the Rawnsley projectors satisfy:

$$P_{\gamma_0(x)} = \rho(\gamma) P_x \rho(\gamma)^{-1} \quad . \tag{3.11}$$

In particular, the square two-point function (3.10) is invariant:

$$\Psi(\gamma_0(x), \gamma_0(y)) = \Psi(x, y)$$

and the Berezin symbol map is $\mathrm{Aut}(X, \omega, L, h)$-equivariant:

$$\sigma(\rho(\gamma) C \rho(\gamma)^{-1})(x) = \sigma(C)(\gamma_0^{-1}(x)) \qquad (C \in \mathrm{End}(E)) \;\;.$$

If we let $\hat{\rho} = \rho^* \otimes_{\mathbb{C}} \rho$ be the representation induced by ρ on $\mathrm{End}(E)$:

$$\hat{\rho}(\gamma)(C) = \rho(\gamma) C \rho(\gamma)^{-1} \quad,$$

then we can write the equivariance property above as follows:

$$\sigma \circ \hat{\rho}(\gamma) = \tau(\gamma_0) \circ \sigma \qquad (\gamma \in \mathrm{Aut}(X, \omega, L, h)) \;\;. \tag{3.12}$$

Here τ is the natural action of $\mathrm{Aut}_{L,h}(X, \omega)$ on $\mathcal{C}^\infty(X)$:

$$\tau(\gamma_0)(f) = f \circ \gamma_0^{-1} \;\;, \tag{3.13}$$

which preserves the symbol space $\Sigma = \mathrm{im}\,\sigma$ as a consequence of (3.12):

$$\tau(\gamma_0)(\Sigma) = \Sigma \;\;.$$

We will sometimes view τ as a representation of $\mathrm{Aut}(X, \omega, L, h)$ via the morphism $\mathrm{Aut}(X, \omega, L, h) \to \mathrm{Aut}_{L,h}(X, \omega)$ (see (2.4)), without indicating this explicitly.

The properties above imply that the Berezin quantization map is equivariant as well:

$$\hat{\rho}(\gamma) \circ Q = Q \circ \tau(\gamma_0) \qquad (\gamma \in \mathrm{Aut}(X, \omega, L, h)) \;\;. \tag{3.14}$$

Finally, the Berezin scalar product satisfies:

$$\prec \tau(\gamma_0)(f), \tau(\gamma_0)(g) \succ_B = \prec f, g \succ_B \qquad (f, g \in \Sigma, \;\; \gamma_0 \in \mathrm{Aut}_{L,h}(X, \omega)) \;\;,$$

which shows that the representation of $\mathrm{Aut}_{L,h}(X, \omega)$ induced by τ on the invariant subspace $\Sigma \subset \mathcal{C}^\infty(X)$ is unitary with respect to $\prec, \;\; \succ_B$. The Berezin trace (3.5) and the Berezin product are also $\mathrm{Aut}_{L,h}(X, \omega)$-invariant:

$$\int \circ \tau(\gamma_0) = \int$$

and:

$$\tau(\gamma_0)(f) \diamond \tau(\gamma_0)(g) = \tau(\gamma_0)(f \diamond g) \;\;.$$

3.3 Changing the scalar product in generalized Berezin quantization

Let us consider what happens when we change the scalar product. An arbitrary Hermitian scalar product $(\ ,\ )'$ on E has the form:

$$(s,t)' = (As, t) = (s, At) \tag{3.15}$$

with A a $(\ ,\ )$-Hermitian positive-definite matrix.[8] The coherent states with respect to the new product $(\ ,\ )'$ are given by:

$$e'_q = A^{-1} e_q \ \ (q \in L_x \setminus \{0\}) \ , \tag{3.16}$$

while the new Rawnsley projectors take the form:

$$P'_x = \frac{1}{\sigma(A^{-1})(x)} A^{-1} P_x \ \ (x \in X) \ . \tag{3.17}$$

The symbol $\sigma(A^{-1})(x) = \frac{(e_q|A^{-1}|e_q)}{(e_q|e_q)}$ of A^{-1} computed with respect to $(\ ,\ )$ and the symbol $\sigma'(A)(x) = \frac{(e'_q|A|e'_q)'}{(e'_q|e'_q)'}$ of A computed with respect to $(\ ,\ )'$ are related by:

$$\sigma(A^{-1})(x) = \frac{1}{\sigma'(A)(x)} \ . \tag{3.18}$$

Notice that $\sigma(A)$ and $\sigma'(A)$ are strictly positive smooth functions on X. Given an operator C, we have more generally:

$$\sigma'(C) = \frac{\sigma(CA^{-1})}{\sigma(A^{-1})} \tag{3.19}$$

and:

$$\sigma(C) = \frac{\sigma'(CA)}{\sigma'(A)} \ . \tag{3.20}$$

Let Q' be the Berezin quantization map defined by $(\ ,\ )'$ and $\Sigma' \subset \mathcal{C}^\infty(X)$ be the image of σ'. Equation (3.19) shows that

$$\Sigma' = \frac{1}{\sigma(A^{-1})} \cdot \Sigma = \left\{ \frac{1}{\sigma(A^{-1})} \cdot f \mid f \in \Sigma \right\}$$

and that:

$$Q'(f) = Q(\sigma(A^{-1})f)A \quad \forall f \in \Sigma' \ .$$

Proposition. The Berezin quantizations defined by two different scalar products on E agree iff the operator A is proportional to the identity, i.e. iff the two scalar products are related by a constant scale factor. In this case, the coherent states differ by a constant homothety and the coherent projectors are equal.

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[8]Of course, A^{-1} and thus A are also Hermitian and strictly positive with respect to $(\ ,\ )'$.

Proof. The quantizations will agree iff $\sigma'(C) = \sigma(C)$ for all $C \in \mathrm{End}(E)$. Using relation (3.20), this implies $\sigma(CA) = \sigma(C)\sigma(A)$ for all $C \in \mathrm{End}(E)$. Taking the complex conjugate and replacing C by $B^\dagger$, we also find $\sigma(AB) = \sigma(A)\sigma(B)$ for all B. Thus $\sigma(AB) = \sigma(BA)$ for all B, which implies that A commutes with all operators on E since σ is injective. Thus $A = \lambda\mathrm{id}_E$ (with $\lambda > 0$) by Schur's Lemma, i.e. the two scalar products differ by a positive constant rescaling. Conversely, it is clear that such a rescaling does not affect the Berezin symbol map. The last statement follows from relations (3.16) and (3.17).

Remarks.

1. The Hermitian conjugate $C^\oplus$ of a linear operator $C \in \mathrm{End}(E)$ with respect to $(\ ,\ )'$ takes the form:
$$C^\oplus = A^{-1}C^\dagger A \ .$$
This allows one to check the identity $(P'_x)^\oplus = P'_x$ by direct computation. The property $(P'_x)^2 = P'_x$ also follows directly from the definition of the symbol $\sigma(A^{-1})$.

2. The operator $\mathcal{I} := A^{1/2}$ is an isometry from the Hilbert space $(E, (\ ,\ ))$ to the Hilbert space $(E, (\ ,\ )')$:
$$(\mathcal{I}u, \mathcal{I}v) = (u, v)' \ . \tag{3.21}$$
In general, this operator is non-local in $x \in X$. The operators $\tilde{P}_x := \mathcal{I}P_x\mathcal{I}^{-1} = A^{1/2}P_x A^{-1/2}$ are orthoprojectors in $(E, (\ ,\ )')$, but they are *not* the Rawnsley projectors of $(\ ,\ )'$. The reason for this is that the coherent states with respect to the latter scalar product have changed, and thus we have to $(\ ,\ )'$-orthoproject onto the *new* coherent states. More generally, any invertible operator $\mathcal{I} \in GL(E)$ can be used to define a new scalar product on E via (3.21). The decomposition $C = UA$ with $A = (\mathcal{I}^\dagger\mathcal{I})^{1/2}$ and $U = \mathcal{I}A^{-1}$ shows that $(u, v)'$ has the form (3.15) and that it depends only on the positive operator A. Of course, the space of scalar products on E can be identified with the homogeneous space $\mathrm{U}(N+1, \mathbb{C})\backslash GL(N+1, \mathbb{C})$. Taking into account the previous proposition, we find that the different Berezin quantizations associated with the line bundle L are parameterized by the points of the homogeneous space $(\mathrm{SU}(N+1, \mathbb{C}) \times \mathbb{C}^*) \backslash GL(N+1, \mathbb{C})$.

3. We can also parameterize the new scalar product $(\ ,\ )'$ by the symbol $a = \sigma(A^{-1})$, i.e. by strictly positive smooth functions $a \in \Sigma$. Then different Berezin quantizations based on L are parameterized by equivalence classes of such functions under rescaling by a positive constant. We have $\Sigma' = \frac{1}{a}\Sigma$ and $Q'(f) = Q(af)Q(a)$ as well as $Q' \circ \Phi = Q$, where the map $\Phi : \Sigma \to \Sigma'$ is given by $\Phi(f) = \frac{\beta(af)}{a}$. Here, β is the Berezin transform of the quantization with respect to the original scalar product $(\ ,\ )$ as defined in section 3.7.

3.4 Integral representations of the scalar product

Let L, $E = H^0(L)$ and $(\ ,\ )$ be as above. Fixing a positive Radon measure μ on X, we consider the problem of representing the scalar product $(\ ,\ )$ on E as the L^2-product

induced by μ and by a Hermitian metric h on the line bundle L. Such representations will be used below when discussing generalized Toeplitz quantization.

Recall that any Hermitian bundle metric h on L can be parameterized by its epsilon function relative to (,):

$$\epsilon(x) := h(x)(q,q)||e_q||^2 = \sum_{i,j=0}^{N} G^{ij} h(x)(s_i(x), s_j(x)) \quad . \tag{3.22}$$

Here, q is any non-zero vector in the fiber L_x, and the right hand side is independent of the choice of q. We use the fact that the value $h(x)$ of the metric on the fiber L_x is uniquely determined by $h(q,q)$. Conversely, any positive smooth function $\epsilon : X \to (0, +\infty)$ determines a Hermitian metric on L via this formula, namely[9] $h(x)(q,q) = \frac{\epsilon(x)}{||e_q||^2}$. This parameterization allows us to describe Hermitian metrics on L through positive smooth functions on X, provided that we have fixed a scalar product (,) on E.

The scalar product (,) on E coincides with the L^2-product induced by μ and h if and only if the following identity holds for any two holomorphic sections s, t of L:

$$(s,t) = \int_X d\mu(x) \; h(x)(s(x), t(x)) \quad .$$

It is easy to see that the right hand side equals $\int_X d\mu(x)\epsilon(x)(s|P_x|t)$. This implies the following:

Proposition. The scalar product (,) on E coincides with the L^2-scalar product induced by (μ, h) iff the relative epsilon function of the pair $(h, (\ ,\ ))$ satisfies the identity

$$\int_X d\mu(x)\epsilon(x)P_x = \mathrm{id}_E \quad , \tag{3.23}$$

i.e. iff the coherent states defined by (,) form an overcomplete set with respect to the measure $\mu_\epsilon = \epsilon\mu$.

Since the Berezin symbol map is injective, equation (3.23) is equivalent with the following Fredholm equation of the first kind:

$$\int_X d\mu(y)\Psi(x,y)\epsilon(y) = 1 \quad (x \in X) \quad .$$

Combining everything, we obtain:

Proposition. There exists a bijection between:

(a) pairs (μ, h) such that μ is a positive Radon measure on X and h is a Hermitian metric on L

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[9]The corresponding metric is well defined since $e_{cq} = \bar{c}^{-1}e_q$ implies $h(x)(cq, cq) = |c|^2 h(x)(q,q)$ for all non-vanishing complex numbers c.

(b) triples $(\mu, (\, , \,), \epsilon)$ such that μ is a positive Radon measure on X, $(\, , \,)$ is a Hermitian scalar product on $H^0(L)$ and ϵ is a non-negative solution of the integral operator equation (3.23), where P_x are the Rawnsley projectors determined by the coherent states defined by $(\, , \,)$.

When the scalar product on E is fixed, equation (3.23) can be viewed as a constraint on the pairs (μ, h) which allow for an integral representation of the scalar product. Taking the trace shows that an epsilon function satisfying (3.23) is normalized to total mass $N+1$ with respect to the measure μ:

$$\int_X d\mu \; \epsilon = N + 1 \quad .$$

In particular, when ϵ is a constant function, then its value must be given by $\epsilon = \frac{N+1}{\mu(X)}$. Also notice that equations (3.2) and (3.23) imply the following integral representation for the trace on $\mathrm{End}(E)$:

$$\mathrm{tr}(C) = \int_X d\mu(x) \; \epsilon(x)\sigma(C)(x) \quad . \tag{3.24}$$

Here σ is the Berezin symbol map defined by the scalar product $(\, , \,)$.

If μ and the scalar product on E are fixed, condition (3.23) can be written in a basis of E as a system of inhomogeneous linear integral equations for ϵ (which in turn determines h):

$$\int_X d\mu \; \epsilon(x)\frac{\overline{\hat{q}(s_i)}\hat{q}(s_j)}{\sum_{i,j=0}^{N} G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)} = G_{ij} \quad . \tag{3.25}$$

Taking the complex conjugate we see that only $\frac{1}{2}(N+1)(N+2)$ of these equations are independent. It is clear that (3.25) admits an infinity of solutions ϵ, so there is an infinity of Hermitian metrics h on L which allow us to represent the scalar product $(\, , \,)$ as an L^2-product with respect to μ. Each such metric h also defines an L^2-scalar product on the space $\Gamma(L)$ of *smooth* sections by formula (2.1), which extends the given scalar product on E. If we let $L^2(\mu, h)$ be the Hilbert space obtained by completing $\Gamma(L)$ with respect to the associated norm, we find an isometric embedding of E into $L^2(\mu, h)$. Hence any solution of (3.25) provides a realization of $H^0(L)$ as a finite-dimensional subspace of an infinite-dimensional Hilbert space.

Remark. When considering quantization with an integral representation of the scalar product on the space of holomorphic sections, one has to deal with three different scalar products on the space of functions. Indeed, the measure μ on X defines a scalar product $\prec , \succ$ on $\mathcal{C}^\infty(X)$:

$$\prec f, g \succ := \int_X d\mu \bar{f}g \quad , \tag{3.26}$$

which extends to the natural scalar product on the space $L^2(X, \mu)$.

On the other hand, the measure $\mu_\epsilon = \mu\epsilon$ appearing in the overcompleteness relation (3.23) defines its own L^2-scalar product $\prec , \succ_\epsilon$ on $\mathcal{C}^\infty(X)$:

$$\prec f, g \succ_\epsilon = \int d\mu \; \epsilon \bar{f}g \quad . \tag{3.27}$$

This extends to the natural scalar product of the space $L^2(X, \mu_\epsilon)$. We have:

$$\prec f, g \succ_\epsilon := \prec f, M_\epsilon g \succ \quad,$$

where $M_\psi : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$, $M_\psi(f) = \psi f$ denotes the operator of multiplication with a smooth function ψ. Notice that M_ψ is $\prec, \ \succ$-Hermitian when ψ is real valued, and $\prec, \ \succ$-positive when ψ is everywhere positive.

Finally, the Berezin symbol space Σ carries the Berezin scalar product $\prec, \ \succ_B$, which is induced from the Hilbert-Schmidt scalar product of $\mathrm{End}(E)$ via the Berezin quantization map (see eq. (3.6)). Hence Σ is endowed with three different scalar products, namely the Berezin product and the restrictions of the products (3.26) and (3.27). It is often the case that various linear operators on Σ are self-adjoint with respect to one of these products but not with respect to the others.

Equation (3.24) provides an integral representation of the Berezin trace:

$$\int f = \mathrm{tr}\, Q(f) = \int_X d\mu(x)\epsilon(x)f(x) \quad (f \in \Sigma) \ , \tag{3.28}$$

which in turns gives the following representation of the Berezin scalar product:

$$\prec f, g \succ_B = \int \bar{f} \diamond g = \langle Q(f), Q(g)\rangle_{\mathrm{HS}} = \int_X d\mu(x)\epsilon(x)(\bar{f} \diamond g)(x) \ . \tag{3.29}$$

3.5 The relative balance condition

Definition. We say that a scalar product on E is μ-*balanced* if equation (3.23) admits the constant solution $\epsilon = \frac{N+1}{\mu(X)}$, i.e. if the following condition is satisfied:

$$\int_X d\mu(x)P_x = \frac{\mu(X)}{N+1}\mathrm{id}_E \ .$$

Hence the scalar product is μ-balanced iff the matrix G satisfies the system of equations:

$$\frac{N+1}{\mu(X)}\int_X d\mu \ \frac{\overline{\hat{q}(s_i)}\hat{q}(s_j)}{\sum_{i,j=0}^N G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)} = G_{ij} \ .$$

For a μ-balanced scalar product, the overcompleteness property of coherent states takes the form $\frac{N+1}{\mu(X)}\int_X d\mu P_x = \mathrm{id}_E$. The Hermitian metric h on L has epsilon function $\epsilon = \frac{N+1}{\mu(X)}$ and therefore is given by:

$$h(x)(q,q) = \frac{N+1}{\mu(X)}\frac{1}{||e_q||^2} = \frac{N+1}{\mu(X)}\frac{1}{\sum_{i,j=0}^N G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)} \ .$$

Let ω_h be the L-polarized Kähler form on X determined by a Hermitian scalar product h on L, and let $\mu_h := \mu_{\omega_h}$ be the Liouville measure on X defined by ω_h. We say that $(\ ,\ )$ is *balanced* if it is μ_h-balanced. This boils down to the system of equations:

$$\frac{N+1}{\mu_h(X)}\int_X d\mu_h \ \frac{\overline{\hat{q}(s_i)}\hat{q}(s_j)}{\sum_{i,j=0}^N G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)} = G_{ij} \ ,$$

where h is determined by:

$$h(x)(q,q) = \frac{N+1}{\mu_h(X)}\frac{1}{\sum_{i,j=0}^N G^{ij}\overline{\hat{q}(s_i)}\hat{q}(s_j)} \ .$$

This is the case considered in [15, 18], which was already mentioned in the previous section.

3.6 Generalized Toeplitz quantization

Let us consider the case when $(\ ,\ )$ is the L^2-scalar product on E determined by a fixed measure μ on X and a Hermitian scalar product h on L. As we saw in the previous section, this can always be achieved by some pair (μ, h). Consider the embedding $E \subset L^2(\mu, h)$. Since the scalar product on $L^2(\mu, h)$ restricts to the original product on E, we will denote it by the same symbol $(\ ,\ )$. Interpreting the coherent vectors e_q as elements[10] of E, the orthogonal projector P_x of E becomes the orthogonal projector Π_x of $L^2(\mu, h)$ onto the one-dimensional subspace of $L^2(\mu, h)$ defined by e_x. Hence equation (3.23) becomes:

$$\int_X d\mu(x)\epsilon(x)\Pi_x = \Pi \ , \tag{3.30}$$

where Π is the orthoprojector of $L^2(\mu, h)$ onto E. We can now define the Toeplitz operator $T(f) \in \mathrm{End}(E)$ associated to a smooth complex function $f \in \mathcal{C}^\infty(X)$:

$$T(f)(s) = \Pi(fs) \ \ \forall s \in E \ . \tag{3.31}$$

Using equation (3.30), we find the integral expression:

$$T(f) = \int_X d\mu(x)\epsilon(x)f(x)P_x \ .$$

The map $T : \mathcal{C}^\infty(X) \to \mathrm{End}(E)$ will be called the *generalized Toeplitz quantization* defined by (L, μ, h). It satisfies $T(\bar{f}) = T(f)^\dagger$ and $T(1_X) = \mathrm{id}_E$. Notice that $T(f)$ depends in an essential manner on the measure $\epsilon\mu$, which is constrained only by condition (3.23). One should contrast this with the generalized Berezin quantization, which is uniquely determined by the scalar product $(\ ,\ )$. Since h characterizes L-polarized Kähler forms, this quantization is useful for an amply polarized Hodge manifold (X, ω, L) endowed with a natural measure μ. Two basic examples are when μ is the Kähler volume form or when X is algebraically Calabi-Yau and μ is the volume form defined by the holomorphic top form.

Behavior under automorphisms. Let us assume that the measure μ and the scalar product $(\ ,\ )$ on E are invariant under the automorphism group of (X, ω, L, h). Then it is easy to see that the relative epsilon function (3.22) is $\mathrm{Aut}_{L,h}(X, \omega)$-invariant:

$$\epsilon(\gamma_0(x)) = \epsilon(x) \qquad (\gamma_0 \in \mathrm{Aut}_{L,h}(X, \omega)).$$

Together with relation (3.11), this shows that generalized Toeplitz quantization is $\mathrm{Aut}(X, \omega, L, h)$-equivariant:

$$T \circ \tau = \hat{\rho} \circ T \ ,$$

i.e.:

$$\rho(\gamma)T(f)\rho(\gamma)^{-1} = T(f \circ \gamma_0^{-1}) \ . \tag{3.32}$$

---

[10]This means that we view the holomorphic sections of L as smooth sections of L, which in particular are elements of $L^2(\mu, h)$.

3.7 Relation between generalized Berezin and Toeplitz quantization

Fixing the scalar product (,), we can consider the associated Berezin quantization as well as any of the Toeplitz quantizations based on an integral representation of this scalar product defined by a compatible pair (μ, h). The relation is given by the *generalized Berezin transform*, the linear map $\beta : \mathcal{C}^\infty(X) \to \Sigma$ defined through:

$$\beta := \sigma \circ T \ ,$$

i.e.:

$$\beta(f)(x) = \int_X d\mu(y) \ \epsilon(y) \Psi(x, y) f(y) \ , \tag{3.33}$$

where Ψ is the squared two-point function (3.10). Adapting an argument of [9], we obtain the following:

Proposition. The linear maps $T : \mathcal{C}^\infty(X) \to \mathrm{End}(E)$ and $\sigma : \mathrm{End}(E) \to \mathcal{C}^\infty(X)$ are adjoint to each other with respect to the scalar product $\prec , \ \succ_\epsilon$ on $L^2(X, \mu_\epsilon)$ and the Hilbert-Schmidt scalar product on $\mathrm{End}(E)$. In particular:

(1) T is surjective since σ is injective.

(2) The Berezin transform $\beta = \sigma \circ T : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ is a $\prec , \ \succ_\epsilon$-non-negative Hermitian operator whose image coincides with Σ.

(3) We have $\ker T = \ker \beta = \Sigma^{\perp_\epsilon} = \{f \in \mathcal{C}^\infty(X) | \prec f, g \succ_\epsilon = 0 \ \ \forall g \in \Sigma\}$ and thus $\beta(\Sigma) \subset \Sigma$ and $T(\Sigma) = \mathrm{End}(E)$.

Proof. We have:

$$\langle T(f), C \rangle_{\mathrm{HS}} = \mathrm{tr}(T(f)^\dagger C) = \mathrm{tr}(T(\bar{f})C) = \int_X d\mu(x)\epsilon(x)\bar{f}(x)\sigma(C) = \prec f, \sigma(C) \succ_\epsilon \ .$$

If C is $\langle , \rangle_{\mathrm{HS}}$-orthogonal to $\mathrm{im}\,T$, it follows that $\prec f, \sigma(C) \succ_\epsilon = 0$ for all $f \in \mathcal{C}^\infty(X)$, which implies $\sigma(C) = 0$. Therefore $C = 0$ by injectivity of σ. The rest is obvious. We also have:

Proposition. (cf. [9]) The Berezin operator $\beta : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ is a contraction with respect to the sup norm $||f||_\infty = \sup_{x \in X} |f(x)|$, i.e.

$$||\beta(f)||_\infty \leq ||f||_\infty \ \ \forall f \in \mathcal{C}^\infty(X) \ .$$

The proof is elementary and virtually identical with that in [9]. It follows that all eigenvalues of β are contained in the interval $[0, 1]$. Notice that β has the form $\beta = \beta|_\Sigma \pi$ where $\beta|_\Sigma$ is a positive contraction in the finite-dimensional subspace Σ and π is the orthoprojector on Σ. Of course, β is also a contraction with respect to the L^2-norm $|| \ ||_\epsilon$ defined by the measure $\mu_\epsilon = \epsilon\mu$.

Since $T = Q \circ \beta$, the Toeplitz quantization of f is related to the Berezin quantization of $\beta(f)$:

$$T(f) = Q(\beta(f)) \ .$$

After restriction to Σ, we have a commutative diagram of bijections:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\;T|_\Sigma\;} & \mathrm{End}(E) \\ \beta|_\Sigma \downarrow & & \| \\ \Sigma & \xrightarrow{\;Q\;} & \mathrm{End}(E) \end{array}$$

where β and T depend on the measure μ_ϵ but Q and Σ depend only on the scalar product $(\;,\;)$. Thus Toeplitz quantizations associated with different L^2-representations of the scalar product $(\;,\;)$ on E give different integral descriptions of the Berezin quantization Q defined by this product. Each Toeplitz quantization is equivalent with Q via the corresponding Berezin transform.

Remark. When μ and $(\;,\;)$ are $\mathrm{Aut}_{L,h}(X,\omega)$ and $\mathrm{Aut}(X,\omega,L,h)$-invariant, respectively, the equivariance properties of σ and T (see eqs. (3.12) and (3.14)) imply that the Berezin map $\beta = \sigma \circ T$ commutes with the natural action of $\mathrm{Aut}_{L,h}(X,\omega)$ on $\mathcal{C}^\infty(X)$:

$$\beta \circ \tau(\gamma_0) = \tau(\gamma_0) \circ \beta \qquad (\gamma_0 \in \mathrm{Aut}_{L,h}(X,\omega)).$$

In this case, the epsilon function is constant and the action τ is unitary on $\mathcal{C}^\infty(X)$ with respect to each of the L^2-scalar products (3.26) and (3.27), while its restriction to Σ is also unitary with respect to the Berezin scalar product.

3.8 The Berezin-Toeplitz lift of linear operators

Recall that the Toeplitz quantization map T is the Hermitian conjugate $\sigma^\dagger$ of the Berezin symbol map with respect to the scalar products $\langle,\;\rangle_{\mathrm{HS}}$ and $\prec,\;\succ_\epsilon$. This implies that the Hermitian conjugate $\sigma^\oplus$ of the symbol map with respect to the scalar products $\langle,\;\rangle_{\mathrm{HS}}$ and $\prec,\;\succ$ is given by:

$$\sigma^\oplus = T \circ M_{\frac{1}{\epsilon}}\;.$$

In particular, we find:

$$\prec \sigma(C_1), \sigma(C_2) \succ = \langle C_1, (\sigma^\oplus \circ \sigma)(C_2)\rangle_{\mathrm{HS}} \qquad (C_1, C_1 \in \mathrm{End}(E))$$

i.e.:

$$\prec f, g \succ = \langle Q(f), \sigma^\oplus(g)\rangle_{\mathrm{HS}} \qquad (f, g \in \Sigma)\;.$$

Let us fix a linear operator $\mathcal{D} : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$.

Definition. The *Berezin-Toeplitz lift* $\hat{\mathcal{D}}$ of $\mathcal{D}$ is the following linear operator on $\mathrm{End}(E)$:

$$\hat{\mathcal{D}} := \sigma^\oplus \circ \mathcal{D} \circ \sigma = T \circ M_{\frac{1}{\epsilon}} \circ \mathcal{D} \circ \sigma : \mathrm{End}(E) \to \mathrm{End}(E)\;.$$

Explicitly, we have:

$$\hat{\mathcal{D}}(C) = \int_X d\mu(x)(\mathcal{D}\sigma(C))(x)P_x \qquad (C \in \mathrm{End}(E))\;. \tag{3.34}$$

The operation of taking the Berezin-Toeplitz lift gives a linear surjection from $\mathrm{End}_{\mathbb{C}}(\mathcal{C}^{\infty}(X))$ to $\mathrm{End}_{\mathbb{C}}(\mathrm{End}(E))$. The identities:

$$\prec \sigma(C_1), \mathcal{D}\sigma(C_2) \succ \; = \langle C_1, \hat{\mathcal{D}}(C_2)\rangle_{\mathrm{HS}} \qquad (C_1, C_2 \in \mathrm{End}(E))$$
$$\prec f, \mathcal{D}(g) \succ \; = \langle Q(f), \hat{\mathcal{D}}(Q(g))\rangle_{\mathrm{HS}} \qquad (f, g \in \Sigma) \qquad (3.35)$$

show that the Berezin-Toeplitz lift is well-behaved with respect to the L^2-product $\prec, \; \succ$ on functions defined by μ. This should be compared with eqs. (3.9) for the Berezin push and pull. In particular, the Berezin-Toeplitz lift $\hat{\mathcal{D}}$ is $\langle, \rangle_{\mathrm{HS}}$-Hermitian when $\mathcal{D}$ is $\prec, \; \succ$-Hermitian and $\langle, \rangle_{\mathrm{HS}}$-positive when $\mathcal{D}$ is $\prec, \; \succ$-positive.

The Berezin-Toeplitz lift of the identity operator $\mathrm{id}_{\mathcal{C}^{\infty}(X)}$ is given by:

$$\nu := \widehat{\mathrm{id}_{\mathcal{C}^{\infty}(X)}} = \sigma^{\oplus} \circ \sigma = T \circ M_{\frac{1}{\epsilon}} \circ \sigma$$

and generally does not coincide with the identity operator on $\mathrm{End}(E)$. The integral expression (3.34) gives:

$$\nu(C) = \int_X d\mu(x) \sigma(C)(x) P_x = \int_X d\mu(x) P_x \mathrm{tr}(P_x C) \; .$$

Define the *modified Berezin transform* $\beta_{\mathrm{mod}} = \beta \circ M_{\frac{1}{\epsilon}} : \mathcal{C}^{\infty}(X) \to \Sigma$ as follows:

$$\beta_{\mathrm{mod}}(f)(x) = \int_X d\mu(y) \Psi(x, y) f(y) \; , \qquad (3.36)$$

where Ψ is the squared two-point function (3.10). This is obtained by formally replacing ϵ with 1 in (3.33). Notice that $\mathrm{im}\, \beta_{\mathrm{mod}} = \Sigma$.

Definition. The *Berezin-Toeplitz transform* $\mathcal{D}_{\diamond} : \Sigma \to \Sigma$ of $\mathcal{D}$ is the Berezin pull (3.8) of $\hat{\mathcal{D}}$:

$$\mathcal{D}_{\diamond} := \hat{\mathcal{D}}_B = \sigma \circ \hat{\mathcal{D}} \circ Q = \beta \circ M_{\frac{1}{\epsilon}} \circ \mathcal{D}|_{\Sigma} = \beta_{\mathrm{mod}} \circ \hat{\mathcal{D}}|_{\Sigma} \; . \qquad (3.37)$$

Explicitly, we have:

$$(\mathcal{D}_{\diamond} f)(x) = \int_X d\mu(y) \Psi(x, y)(\mathcal{D}f)(y) \qquad (f \in \Sigma) \; .$$

The last equation in (3.35) shows that the bilinear form of $\mathcal{D}_{\diamond}$ with respect to the Berezin scalar product equals the bilinear form of $\mathcal{D}$ with respect to the L^2-scalar product induced by μ:

$$\prec f, \mathcal{D}(g) \succ \; = \prec f, \mathcal{D}_{\diamond}(g) \succ_B \quad (f, g \in \Sigma) \; .$$

In particular, $\mathcal{D}_{\diamond}$ is $\prec, \; \succ_B$-Hermitian iff $\mathcal{D}$ is $\prec, \; \succ$-Hermitian and $\prec, \; \succ_B$-positive iff $\mathcal{D}$ is $\prec, \; \succ$-positive. The Berezin-Toeplitz transform $\mathrm{id}_{\diamond} = \nu_B$ of the identity operator $\mathrm{id}_{\mathcal{C}^{\infty}(X)}$ coincides with the modified Berezin transform:

$$\mathrm{id}_{\diamond} = \beta_{\mathrm{mod}} \; .$$

3.9 Changing the scalar product in generalized Toeplitz quantization

Let us consider what happens when we change the scalar product on E while keeping h and μ fixed. Equations (3.17) and (3.23) give:

$$\int_X d\mu(x)\epsilon(x)\sigma(A^{-1})(x)P'_x = A^{-1} \ , \tag{3.38}$$

i.e.:

$$(s,t) = \int_X d\mu(x)\epsilon(x)\sigma(A^{-1})(x)\frac{(s|e'_q)'(e'_q|t)'}{(e'_q|e'_q)'} \ ,$$

where the scalar product (s,t) on the left hand side is the original (unprimed) product. Relations (3.15), (3.16) and (3.22) show that the epsilon function of the pair $(h,(\ ,\ )')$ is given by:

$$\epsilon'(x) = \epsilon(x)\sigma(A^{-1}) \ , \tag{3.39}$$

so (3.38) takes the form:

$$\int_X d\mu(x)\epsilon'(x)P'_x = A^{-1} \ ,$$

which allows us to express the original scalar product as:

$$(s,t) = (s, A^{-1}t)' = \int_X d\mu(x)\epsilon'(x)(s|P'_x|t)' = \int_X d\mu(x)\epsilon'(x)\frac{\overline{\hat{q}(s)}\hat{q}(t)}{(e'_q|e'_q)'} \ .$$

The original epsilon function can be recovered from equations (3.39) and (3.18):

$$\epsilon(x) = \sigma'(A)\epsilon'(x) \ ,$$

while the original Rawnsley coherent states can be recovered as $e_q = Ae'_q$. In principle, this allows us to recover the original Toeplitz quantization from knowledge of $(\ ,\ )'$. We can define an operator:

$$T'(f) := \int_X d\mu(x)\epsilon'(x)f(x)P'_x \ ,$$

which satisfies $T'(f)^{\oplus} = T(\bar{f})$ as well as:

$$\mathrm{tr}(AT'(f)) = \int_X d\mu(x)\epsilon(x)f(x) = \mathrm{tr}(T(f))$$

and:

$$T'(1_X) = A^{-1} \ .$$

In practice, one is often interested in the case when $(\ ,\ )$ is the L^2-scalar product $\langle\ ,\ \rangle_1^{\mu,h}$ defined by μ and h:

$$(s,t) := \langle s,t \rangle_1^{\mu,h} = \int_X d\mu(x)h(x)(s(x),t(x)) \ .$$

In such a situation one might be able to compute the coherent states and epsilon function with respect to another scalar product $(\ ,\ )'$ on E. Then the expressions above allow one to recover the Toeplitz quantization with respect to the L^2-scalar product of (μ,h).

We can also ask about relating the Berezin quantization Q' defined by $(\ ,\ )'$ to the Toeplitz quantization defined by the L^2-scalar product $\langle\ ,\ \rangle_1^{\mu,h}$. The two quantizations are related by the map $\beta := \sigma' \circ T : \mathcal{C}^\infty(X) \to \Sigma'$:

$$T(f) := Q'(\beta(f)) \ \ .$$

We have the integral expression

$$\beta(f)(x) = \int_X d\mu(y)\epsilon(y)f(y)\lambda(x,y) \ \ ,$$

where $\lambda(x,y) = \sigma'(P_y)(x) = \frac{\sigma(A^{-1}P_y)(x)}{\sigma(A^{-1})(x)} = \frac{\mathrm{tr}(A^{-1}P_yP_x)}{\mathrm{tr}(A^{-1}P_x)} = \frac{\sigma(A^{-1}(y))}{\sigma(A^{-1}(x))}\sigma(P'_y(x))$. Notice that $\lambda(x,y)$ need not equal $\lambda(y,x)$.

3.10 Extension to powers of L

We can easily extend everything by replacing the very ample line bundle L with any of its positive powers $L^k := L^{\otimes k}$ ($k \geq 1$). In this case, generalized Berezin quantization requires a choice of Hermitian scalar products $(\ ,\ )_k$ on each of the finite-dimensional vector spaces $E_k := H^0(L^k)$. Accordingly, we have coherent states $e_x^{(k)} \in E_k$ and Rawnsley projectors $P_x^{(k)}$, as well as surjective Berezin symbol maps $\sigma_k : \mathrm{End}(E_k) \to \mathcal{C}^\infty(X)$ whose images we denote by Σ_k. The inverse of the corestrictions $\sigma_k|^{\Sigma_k}$ define a sequence of Berezin quantization maps $Q_k : \Sigma_k \to \mathrm{End}(E_k)$. The entire construction depends crucially on the precise sequence of Hermitian scalar products $(\ ,\ )_k$ chosen for the spaces E_k.

4. Classical Berezin and Toeplitz quantization of compact Hodge manifolds

Classical Berezin and Toeplitz quantization of prequantized Hodge manifolds (X,ω,L,h) arises as a particular case of the generalized constructions discussed above. In the classical set-up, one fixes an integer k_0 such that L^{k_0} is very ample. For each integer $k \geq k_0$, we apply the general construction for the ample line bundle L^k endowed with the Hermitian scalar product $h_k := h^{\otimes k}$ and with the L^2-scalar product $\langle\ ,\ \rangle_k$ on $E_k := H^0(L^k)$ induced by h_k and by the Liouville measure μ_ω defined by ω, see eq. (2.2).

4.1 Classical Berezin and Toeplitz quantization

Applying Berezin quantization with the choices above leads to bijective symbol maps $\sigma_k : E_k \to \Sigma_k \subset \mathcal{C}^\infty(X)$ and quantization maps $Q_k : \Sigma_k \to \mathrm{End}(E_k)$. We also have surjective Toeplitz quantization maps $T_k : \mathcal{C}^\infty(X) \to \mathrm{End}(E_k)$ given by $T_k(f) := \Pi_k(f\cdot)$, where $\Pi_k : L_k^2(L,h,\mu_\omega) \to E_k$ is the orthoprojector on the subspace of holomorphic sections. The surjective Berezin transforms $\beta_k = \sigma_k \circ T_k : \mathcal{C}^\infty(X) \to \Sigma_k$ relate the two quantizations via $T_k = Q_k \circ \beta_k$. The maps β_k and T_k have equal kernel $\Sigma_k^{\perp_{\epsilon_k}} \subset \mathcal{C}^\infty(X)$ and the restrictions of β_k and T_k give linear isomorphisms. For each $k \geq k_0$, we have a commutative diagram of bijections:

$$
\begin{array}{ccc}
\Sigma_k & \xrightarrow{T_k|_\Sigma} & \mathrm{End}(E_k) \\
\beta_k|_\Sigma \downarrow & & \| \\
\Sigma_k & \xrightarrow{Q_k} & \mathrm{End}(E_k)
\end{array}
$$

where $\beta_k|_{\Sigma_k}$ is a strictly positive self-adjoint contraction.

It is convenient to consider the Hilbert space:

$$\mathcal{E}_X := \overline{\oplus}_{k=0}^{\infty}(E_k, \langle\,,\,\rangle_k) \quad . \tag{4.1}$$

Recall that the completed direct sum $\mathcal{E}_X$ consists of all infinite sequences $s = \sum_{k=0}^{\infty} s_k$ with $s_k \in E_k$ which satisfy the condition:

$$\sum_{k=0}^{\infty} \langle s_k, s_k \rangle_k < \infty \quad .$$

It is endowed with the scalar product:

$$\langle s, t \rangle_X = \sum_{k=0}^{\infty} \langle s_k, t_k \rangle_k \quad . \tag{4.2}$$

Then the E_k become closed subspaces of $\mathcal{E}_X$ and the scalar products $\langle s_k, t_k \rangle_k$ coincide with the restriction of $\langle\,,\,\rangle_X$.

Similarly, we let $H_k := L_k^2(L^k, \mu, h)$ be the L^2-completion of $\Gamma(L^k)$ and $\mathcal{H}_X := \overline{\oplus}_{k=0}^{\infty} H_k$ be the Hilbert direct sum of H_k. Then $\mathcal{E}_X$ is a closed subspace in $\mathcal{H}_X$ and the orthoprojector Π on the former decomposes as:

$$\Pi = \sum_{k=0}^{\infty} \Pi_k \quad , \tag{4.3}$$

where Π_k is the orthoprojector on E_k inside H_k. The projector (4.3) is sometimes called the *Szegö projector*.

Remark. Consider the total space $\mathbb{S}$ of the unit circle bundle of L^*. This is a Cauchy-Riemann (CR) manifold of CR-codimension one, whose CR-structure is induced by its obvious embedding as a real hypersurface in the total space $\mathbb{L}$ of L^*. Moreover, it is the boundary of the total space $\mathbb{D}$ of the unit disk bundle of L^*, which is known to be a strictly pseudoconvex domain in $\mathbb{L}$. The Kähler form ω of X induces a contact one-form α on $\mathbb{S}$ such that the pull-back of ω through the projection of the circle bundle equals $d\alpha$. The *Hardy space* is the Hilbert space of all CR-holomorphic functions on $\mathbb{S}$, endowed with the L^2-scalar product induced by the volume form $\alpha \wedge (d\alpha)^{\dim X}$ of the contact form α. This is the usual Hardy space of boundary values of holomorphic functions on the domain $\mathbb{D}$. It is well-known that the Hardy space is isometric to the Hilbert space $\mathcal{E}_X$. Because of this, we will identify the two and sometimes refer to the latter as the Hardy space. Similarly, the space of L^2-functions on $\mathbb{S}$ identifies with $\mathcal{H}_X$.

4.2 The formal star products and associated deformation quantizations

Let $\{\,,\,\}$ be the Poisson bracket defined by ω, h be a formal variable and consider the $\mathbb{C}[[h]]$-module $\mathcal{C}^{\infty}(X)[[h]]$ of formal power series with smooth function coefficients. Recall that a normalized formal star product on X is a $\mathbb{C}[[h]]$-bilinear map $\star : \mathcal{C}^{\infty}(X)[[h]] \times \mathcal{C}^{\infty}(X)[[h]] \to \mathcal{C}^{\infty}(X)[[h]]$ on this module such that:

(a) $\star$ is associative

(b) The coefficients of the formal expansion:[11]

$$f \star g = \sum_{n=0}^{\infty} h^j C_j(f,g) \quad (f,g \in \mathcal{C}^{\infty}(X))$$

are bi-differential operators satisfying the identities:

1. $C_0(f,g) = fg$
2. $C_1(f,1) = C_1(1,f) = 0$
3. $C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\}$

Notice that we use an expansion in h rather than $\hbar = \frac{h}{2\pi}$ due to our convention for integral symplectic forms ($[\omega] \in H^2(X,\mathbb{Z})$), as required by agreement with the Bohr-Sommerfeld condition.

The Toeplitz deformation quantization. It was shown in [10] that there exists a unique normalized formal star product $\star_T$ on X (known as the Toeplitz star product) whose coefficients C_j have the property:

$$||T_k(f)T_k(g) - \sum_{j=0}^{N} \frac{1}{k^j} T_k(C_j(f,g))||_k = K_p(f,g)\frac{1}{k^N}$$

for all m and all sufficiently large k. Here, $|| \; ||_k$ is the operator norm on $\mathrm{End}(E_k)$ and the $K_p(f,g)$ are constants which depend only on p and f,g. This can be interpreted as an asymptotic expansion:

$$T_k(f)T_k(g) \sim \sum_{j=0}^{\infty} \frac{1}{k^j} T_k(C_j(f,g)) \;\; \text{for} \;\; k \to \infty \; , \tag{4.4}$$

where the right hand side formally corresponds to $T_k(f \star_T g)$ at $h = \frac{1}{k}$ (here $h = 2\pi\hbar$). It should be stressed that the formal star product $\star_T$ captures the entire asymptotic expansion (4.4), which includes information from *all* values of $k \geq k_0$. The Toeplitz star product has 'anti-separation of variables' in the sense that $a \star_T g = f \star_T b = 0$ whenever a is antiholomorphic and b is holomorphic.

Remark. One has [6]:

$$||T_k(f)T_k(g) - T_k(fg)||_k = O\left(\frac{1}{k}\right) \;\; \text{for} \;\; k \to \infty$$

as well as:

$$||ik[T_k(f), T_k(g)] - T_k(\{f,g\})||_k = O\left(\frac{1}{k}\right) \;\; \text{for} \;\; k \to \infty$$

---

[11]Notice that these completely determine the star product.

and:

$$||f||_\infty - \frac{C_f}{k} \leq ||T_k(f)||_k \leq ||f||_\infty$$

for all $f \in \mathcal{C}^\infty$ and some constant C_f depending only on f. In particular, one finds $\lim_{k\to\infty} ||T_k(f)|| = ||f||_\infty$. These properties imply that the continuous field of C^*-algebras on the set $I = \{\frac{1}{k} | k \in \mathbb{N}^*\} \cup \{0\}$ given by $A_{\frac{1}{k}} := (\Sigma_k, || \ ||_k)$, with $A_0 := (\mathcal{C}^\infty(X), || \ ||_\infty)$ and section $\frac{1}{k} \to T_k$, $T_0 := \mathrm{id}_{\mathcal{C}^\infty(X)}$ forms a 'strict quantization' in the sense of Rieffel [22] though not a 'strict deformation quantization'.

The Berezin deformation quantization. It was further shown in [12] that the Berezin transform β_k has an asymptotic expansion $\beta_k \sim \sum_{r=0}^\infty \frac{1}{k^r}\beta_r$ with $\beta_0 = 1$. This allows one to define an automorphism of the $\mathbb{C}[[h]]$-module $\mathcal{C}^\infty(X)[[h]]$, known as the *formal Berezin transform*, via:

$$\boldsymbol{\beta} = \sum_{r=0}^\infty \beta_r h^r \ . \tag{4.5}$$

The Berezin star product $\star_B$ is the formal normalized star product obtained from $\star_T$ via the formal Berezin transform, $f \star_B g = \boldsymbol{\beta}(\boldsymbol{\beta}^{-1}(f) \star_T \boldsymbol{\beta}^{-1}(g))$. Again, it should be stressed that $\star_B$ contains information from all powers L^k, $k \geq k_0$. We have the relation:

$$\boldsymbol{\beta}(f \star_T g) = \boldsymbol{\beta}(f) \star_B \boldsymbol{\beta}(g) \ .$$

The Berezin star product has 'separation of variables' in the sense of [11], i.e. one has $a \star_B g = f \star_B b = 0$ whenever a and b are holomorphic and antiholomorphic functions, respectively [8].

4.3 Relation with geometric quantization

In Konstant-Souriau geometric quantization of Kähler manifolds [21], one considers the linear maps $\Theta_k : \mathcal{C}^\infty(X) \to \mathrm{End}(E_k)$ given by:

$$\Theta_k(f) := \Pi_k \theta_k(f) \ , \tag{4.6}$$

where

$$\theta_k(f) := -i\nabla^{(k)}_{X_f^k} - f \cdot \quad .$$

X_f is the Hamiltonian vector field defined by the smooth function f with respect to the symplectic form $k\omega$ and $\nabla^{(k)}$ is the Chern connection on L^k. This procedure corresponds to using the so-called *complex polarization*. One has the following relation [25]:

$$\Theta_k(f) = T_k \left(f - \frac{1}{2k}\Delta f \right) \quad \forall f \in \mathcal{C}^\infty(X) \ ,$$

where Δ is the Laplace operator of (X, ω) — at least on compact symmetric spaces. Notice that we use conventions in which $\Theta_k(f)^\dagger = \Theta_k(\bar{f})$.

4.4 The Berezin product or coherent state star product

In the vast majority of the literature on fuzzy geometry, the "star product" used is the Berezin product $\diamond_k : \Sigma_k \times \Sigma_k \to \Sigma_k$ introduced in section 3.2:

$$f \diamond_k g := \sigma_k(Q_k(f)Q_k(g)), \quad f,g \in \Sigma_k \ . \tag{4.7}$$

This product is also called the *coherent state star product*, as $\sigma_k(C) = \mathrm{tr}(CP_x^{(k)})$ is determined by the coherent states. It is associative by definition and the algebra $(\Sigma_k, \diamond_k, {}^-)$ is isomorphic as a $*$-algebra to $(\mathrm{End}(E_k), \circ, \dagger)$ with the Berezin quantization map Q_k providing the isomorphism.

Note that the Berezin product is *not* a formal star product as it is defined only on Σ_k, instead of $\mathcal{C}^\infty(X)[[h]]$. However, it has been shown [5] that in the case of flag manifolds, there is a formal differential star product on the set $\Sigma_\bullet := \bigcup_{k=0}^\infty \Sigma_k$, which agrees with the asymptotic expansion of the Berezin products on Σ_k for certain $h = \frac{1}{k}$.

As an example, consider the Berezin quantization of $(\mathbb{P}^n, \omega_{\mathrm{FS}})$ with the prequantum line bundle H^k, where H is again the hyperplane line bundle. If we normalize the homogeneous coordinates on $\mathbb{P}^n$ by demanding that $|z| = 1$, we obtain the particularly simple form [28]:

$$f \diamond_k g = \sum_{i_1,\dots,i_k} \left(\frac{1}{k!} \frac{\partial}{\partial z_{i_1}} \cdots \frac{\partial}{\partial z_{i_k}} f \right) \left(\frac{1}{k!} \frac{\partial}{\partial \bar{z}_{i_1}} \cdots \frac{\partial}{\partial \bar{z}_{i_k}} g \right) \ .$$

A different form of the Berezin product corresponding to a finite sum resembling the first terms in an expansion of a formal star product can be written down in the real setting, using the embedding $\mathbb{P}^n \hookrightarrow \mathbb{R}^{(n+1)^2-1}$ [17].

4.5 The quantization of affine spaces

Classical Berezin-Toeplitz quantization can be extended to the non-compact case[12] upon replacing the space of holomorphic sections of the quantum line bundle L with the subspace of those holomorphic sections which are square integrable with respect to an appropriately weighted version of the Liouville measure $\frac{\omega^n}{n!}$. In particular, this can be applied to the case of complex affine spaces, where the weight is provided by the global Kähler potential, leading to the well-known construction of the Bargmann representation of the bosonic Fock space. In this subsection, we recall this construction in order to fix our notations for later use.

Let us start with a few remarks about the coordinate-free description. If V is an $n+1$-dimensional complex vector space, the algebra $B := \mathbb{C}[E]$ of polynomial functions over the dual space $E := V^* = \mathrm{Hom}_\mathbb{C}(V, \mathbb{C})$ is the symmetric algebra associated with V:

$$B = \mathbb{C}[E] = \oplus_{k=0}^\infty E^{\odot k} = \oplus_{k=0}^\infty (V^*)^{\odot k} \ . \tag{4.8}$$

We let $B_k := E^{\odot k} \subset B$. As an algebraic variety, the affine space over V is the affine spectrum $A(V) = \mathrm{Spec}B$ of this algebra. A choice of basis $e_0 \dots e_n$ for V allows us to define coordinate functionals $z_j \in E = \mathrm{Hom}_\mathbb{C}(V, \mathbb{C})$ such that $z_j(v) = v_j$ for all $v =$

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[12]In fact, historically this was the original class of examples.

$\sum_{j=0}^{n} v_j e_j \in V$. Thus (z_j) is the basis of E dual to the given basis of V. Once a basis of V has been fixed, we can write the elements of B as polynomial functions over V:

$$f = \sum_{|\mathbf{p}|=\text{bounded}} a_{\mathbf{p}} \chi_{\mathbf{p}} \ ,$$

where $\mathbf{p} = (p_0 \dots p_n) \in \mathbb{N}^{n+1}$, $|\mathbf{p}| := \sum_{j=0}^{n} p_j$ and:

$$\chi_{\mathbf{p}} := z^{\mathbf{P}} := z_0^{p_0} \dots z_n^{p_n} \quad . \tag{4.9}$$

We denote the symmetrized tensor product $\odot$ by juxtaposition. As a function on V, we have:

$$f(v) = \sum_{|\mathbf{p}|} a_{\mathbf{p}} v_0^{p_0} \dots v_n^{p_n} \quad .$$

If we use the given basis to identify V with $\mathbb{C}^{n+1}$, then v identifies with the vector $(v_0 \dots v_n)$ and we obtain the polynomial function $f(v_0 \dots v_n) = \sum_{|\mathbf{p}|} a_{\mathbf{p}} v_0^{p_0} \dots v_n^{p_n}$ on $\mathbb{C}^{n+1}$. In this case, B identifies with the polynomial algebra $\mathbb{C}[v_0 \dots v_n]$ in $n+1$ variables, which is the coordinate ring of $\mathbb{C}^{n+1}$.

Given a Hermitian scalar product $(\ ,\ )$ on E, we have an induced product on V (denoted by the same symbol) and can chose the basis $e_0 \dots e_n$ to be orthonormal with respect to this induced product. In this case, the basis $z_0 \dots z_n$ of E is also orthonormal and the scalar product of two elements of V can be written as:

$$(u,v) = \bar{z}(u) \cdot z(v) = \sum_{j=0}^{n} \overline{z_j(u)} z_j(v) = \sum_{j=0}^{n} \overline{u_j} v_j \ ,$$

where $z(v) := (z_0(v) \dots z_n(v)) = (v_0 \dots v_n)$. Notice that $||v||^2 = |z(v)|^2 = \sum_{j=0}^{n} |z_j(v)|^2$. The scalar product induces a flat Hermitian metric on V whose Kähler form:

$$\omega = \frac{i}{2\pi} \sum_{j=0}^{n} dz_j \wedge d\bar{z}_j$$

corresponds to the standard symplectic form on the underlying real vector space $V_{\mathbb{R}}$ of V if we set $z_j = \frac{1}{\sqrt{2}}(q_j + ip_j)$, where $q_j, p_j \in \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ are real coordinates on $V_{\mathbb{R}}$:

$$\omega = \frac{1}{2\pi} \sum_{j=0}^{n} dq_j \wedge dp_j \quad .$$

Since $\frac{\omega^{n+1}}{(n+1)!} = \frac{1}{(2\pi)^{n+1}} dq_0 \wedge dp_0 \wedge \dots \wedge dq_n \wedge dp_n$, the associated Liouville measure is the scaled Lebesgue measure $d\mu = \frac{1}{(2\pi)^{n+1}} d^{n+1}q \, d^{n+1}p$ on $V_{\mathbb{R}}$. The Kähler form is polarized with respect to the trivial line bundle $\mathcal{O} = V \times \mathbb{C}$, whose unit section we denote by $s_0 = 1$ (this is just the unit constant function on V). $\mathcal{O}$ becomes a quantum line bundle when endowed with the Hermitian metric h given by:

$$\hat{h}(v) := h(v)(s_0(v), s_0(v)) := e^{-|z(v)|^2} = e^{-||v||^2} \quad .$$

The unit section gives the global Kähler potential:

$$K(v) = -\log \hat{h}(v) = |z(v)|^2 = ||v||^2 \quad .$$

The holomorphic sections of $\mathcal{O}$ are simply the entire functions f on V, since every such section can be written as $s_f = f s_0$. The L^2-scalar product (subsequently to be referred to as the *Bargmann product*) is:

$$\langle f, g \rangle_B := \langle s_f, s_g \rangle = \int_V d\mu(v) e^{-|z(v)|^2} \bar{f}(v) g(v) = \int_V d\nu(v) \bar{f}(v) g(v) \quad , \tag{4.10}$$

where:

$$d\nu(v) = e^{-|z(v)|^2} d\mu(v) = \frac{1}{(2\pi)^{n+1}} e^{-\frac{1}{2} \sum_{j=0}^{n} (q_j^2 + p_j^2)} \ d^{n+1}q d^{n+1}p$$

is the weighted Lebesgue measure, which is normalized to unit total mass:

$$\int_V d\nu = 1 \quad .$$

The space of square integrable holomorphic sections of $\mathcal{O}$ is the well-known weighted Bargmann space $\mathcal{B}(V) := L^2_{\text{hol}}(V, d\nu)$ of ν-square integrable entire functions on V, which contains the algebra $B = \mathbb{C}[E]$ of polynomial functions as a dense subspace. The Bargmann space carries the unitary representation of the $n + 1$-th Weyl group with creation and annihilation operators given by:

$$(\hat{a}_i^\dagger f)(v) := z_i(v) f(v) \quad , \quad (\hat{a}_i f)(v) = \partial_i f(v) \quad ,$$

where $\partial_i = \frac{\partial}{\partial e_i}$ is the directional derivative along e_i. We have:

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \quad .$$

The normalized vacuum vector is the constant unit function $|0\rangle := 1$. For every tuple $\mathbf{p} = (p_0 \ldots p_n) \in \mathbb{N}^{n+1}$, let $\mathbf{p}! := p_0! \ldots p_n!$. We have:

$$||\chi_\mathbf{p}||_B = \sqrt{\mathbf{p}!} \quad , \tag{4.11}$$

where $\chi_\mathbf{p}$ are the monomials (4.9). The normalized occupation vectors are given by:

$$|\mathbf{p}\rangle = \frac{1}{\sqrt{\mathbf{p}!}} \chi_\mathbf{p} = \frac{(\hat{a}^\dagger)^\mathbf{p}}{\sqrt{\mathbf{p}!}} |0\rangle \quad . \tag{4.12}$$

They are the common eigenvectors of the particle number operators $\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i$:

$$\hat{N}_i |\mathbf{p}\rangle = p_i |\mathbf{p}\rangle \quad .$$

An entire function:

$$f = \sum_{\mathbf{p} \in \mathbb{N}^{n+1}} c_\mathbf{p} \chi_\mathbf{p} = \sum_{\mathbf{p} \in \mathbb{N}^{n+1}} c_\mathbf{p} \sqrt{\mathbf{p}!} |\mathbf{p}\rangle \quad (c_\mathbf{p} \in \mathbb{C}) \tag{4.13}$$

belongs to $\mathcal{B}$ iff

$$\langle f|f\rangle_B = \sum_{\mathbf{p}\in\mathbb{N}^{n+1}} \mathbf{p}!|c_{\mathbf{p}}|^2 < \infty \ \ .$$

Defining the total particle number operator $\hat{N} = \sum_{i=1}^n \hat{N}_i$, relation (4.12) shows that its eigenspace of eigenvalue k coincides with B_k. We have the orthogonal decomposition $\mathcal{B} = \overline{\oplus}_{k=0}^\infty B_k$ (completed direct sum) with $B_k = \ker(\hat{N} - k)$. It is easy to see that $\mathcal{B}$ is unitarily isomorphic with the bosonic Fock space $\mathcal{F}_s(E) = \overline{\oplus}_{k=0}^\infty E^{\odot k}$ over the finite-dimensional Hilbert space $(E, (\ ,\ ))$. Under this identification, $|\mathbf{p}\rangle$ becomes the orthonormal basis of the Fock space canonically associated with the orthonormal basis $(z_0 \ldots z_n)$ of E.

Since $\mathcal{O}$ has a global nowhere vanishing section (the unit section s_0), we can consider Rawnsley's coherent vectors with respect to $q = s_0(v) = 1 \in \mathcal{O}_v$. These are the usual Glauber vectors:

$$|v\rangle = e^{\sum_{i=0}^n \bar{v}_i \hat{a}_i^\dagger}|0\rangle = \sum_{\mathbf{p}} \frac{\overline{v}^{\mathbf{p}}}{\sqrt{\mathbf{p}!}}|\mathbf{p}\rangle \ \ ,$$

where $\overline{v}^{\mathbf{p}} = \overline{v_0}^{p_0} \ldots \overline{v_n}^{p_n}$. One has the well-known identity:

$$f(v) = \ \langle v|f\rangle_B \quad (f \in \mathcal{B}) \ \ .$$

We have:

$$\hat{a}_i|v\rangle = \bar{v}_i|v\rangle \ \ , \quad \langle u|v\rangle_B = e^{(v,u)} \ \ .$$

In particular $|v\rangle$ has norm $e^{||v||^2}$. The reproducing kernel is the well-known Bergman kernel:

$$K_B(u,v) = \frac{\langle u|v\rangle}{\sqrt{\langle u|u\rangle \langle v|v\rangle}} = e^{-\frac{1}{2}(|u|^2+|v|^2)+(v,u)} \ \ .$$

The Rawnsley projector is[13] $P_v = \frac{1}{\langle v|v\rangle_B}|v\rangle\langle v|_B = e^{-||v||^2}|v\rangle\langle v|_B$. The epsilon function is constant and equal to one:

$$\epsilon_{\mathbb{C}^{n+1}} = \hat{h}(v)\langle v|v\rangle_B = 1 \ \ .$$

The decomposition of the identity takes the form:

$$\int_V d\mu(v) P_v = 1 \Leftrightarrow \int_V d\mu(v) e^{-||v||^2}|v\rangle\langle v|_B = 1 \Leftrightarrow \int_V d\nu(v)|v\rangle\langle v|_B = 1 \ \ .$$

Toeplitz quantization of $A(V)$. The Toeplitz quantization of $f \in \mathcal{C}^\infty(V,\mathbb{C})$ is given by:

$$T(f) = \int_V d\mu(v) f(v) P_v = \int_V d\mu(v) e^{-||v||^2} f(v)|v\rangle\langle v|_B \ \ . \tag{4.14}$$

In particular, we have $T(z_i) = \hat{a}_i^\dagger$ and $T(\bar{z}_i) = \hat{a}_i$. When f is a polynomial in z and $\bar{z}$, (4.14) obviously reduces to the anti-Wick prescription:

$$T(f) = {\vdots}\,f(\hat{a}^\dagger, \hat{a})\,{\vdots} \ \ ,$$

where the triple dots indicate antinormal ordering. In this case, T is not surjective due to the infinite-dimensionality of the Bargmann space.

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[13]Recall that $\langle v|_B$ stands for the linear functional $\langle v|_B(\psi) := \langle v|\psi\rangle_B$ which is Riesz dual to the vector $|v\rangle$. Since Riesz duality depends on the choice of scalar product on $\mathcal{B}$, we use an underscript B on bra vectors.

Berezin quantization of $A(V)$**.** The Berezin symbol map $\sigma : \mathcal{L}(\mathcal{B}) \to \mathcal{C}^\infty(V, \mathbb{C})$ is defined on the algebra $\mathcal{L}(\mathcal{B})$ of bounded operators in the Bargmann space. The symbol of a bounded operator C takes the form:

$$\sigma(C)(v) = e^{-||v||^2} \langle v | C | v \rangle_B \ ,$$

while the Berezin transform $\beta(f) = \sigma \circ T$ is given by:

$$\beta(f)(u) = \int_V d\mu(v) f(v) e^{-||u-v||^2} \ .$$

Thus $\beta = \frac{1}{2^{n+2}} e^{-4\Delta}$, where Δ is the Laplacian on $V_\mathbb{R}$; this is the heat kernel up to normalizations.

The symbol map gives rise to the Berezin quantization $Q : \Sigma \to \mathcal{L}(\mathcal{B})$, where $\Sigma \subset \mathcal{C}^\infty(\mathbb{C}^{n+1})$ is the image of σ. We have $Q(z_i) = \hat{a}_i^\dagger$ and $Q(\bar{z}_i) = \hat{a}_i$. For a polynomial function $f(z, \bar{z})$, we find:

$$Q(f) =: f(\hat{a}^\dagger, \hat{a}): \ ,$$

where the double dots indicate normal ordering. Hence both quantization prescriptions send z_i into $\hat{a}_i^\dagger$ and $\bar{z}_i$ into $\hat{a}_i$, but Toeplitz quantization corresponds to anti-Wick ordering, while Berezin quantization corresponds to Wick ordering.

Truncated coherent vectors. For later use, consider the expansion of Glauber's coherent vectors in components of fixed total particle number:

$$|v\rangle = \sum_{k=0}^\infty |v, k\rangle \ ,$$

where the 'truncated coherent vectors'

$$|v, k\rangle := \frac{1}{k!} \left(\sum_{i=0}^n \bar{v}_i \hat{a}_i^\dagger \right)^k |0\rangle = \sum_{|\mathbf{p}|=k} \frac{\bar{v}^{\mathbf{p}}}{\sqrt{\mathbf{p}!}} |\mathbf{p}\rangle$$

satisfy

$$\hat{N}|v, k\rangle = k|v, k\rangle$$
$$\langle u, k | v, l \rangle_B = \delta_{k,l} \frac{1}{k!} [(v, u)]^k \ .$$

In particular, we have $\langle v, k | v, k \rangle_B = \frac{1}{k!} ||v||^{2k}$. Since $[\hat{N}, \hat{a}_i] = -1$ and $[\hat{N}, \hat{a}_i^\dagger] = +1$, we find:

$$\hat{a}_i |v, k\rangle = \bar{z}_i |v, k - 1\rangle \ . \tag{4.15}$$

Notice that $|\lambda v, k\rangle = \bar{\lambda}^k |v, k\rangle$ for any $\lambda \in \mathbb{C}$, and therefore the ray $\mathbb{C}^* |\lambda v, k\rangle$ depends only on the image $[v]$ of v in the projective space $\mathbb{P}V = (V \setminus \{0\})/\mathbb{C}^*$.

4.6 The quantization of complex projective spaces

We next consider the case of complex projective spaces, which has been studied extensively in the literature on Berezin quantization. Using the 'truncated coherent vectors' of the previous subsection, we will show that the (yet to be defined) Berezin-Bergman quantization of $\mathbb{P}^n$ coincides with its Berezin quantization. In particular, the fuzzy version of $\mathbb{P}^n$ considered in [1] coincides with its well-known Berezin quantization.[14] To make contact with the formalism used in [1], we will use the fact that the homogeneous coordinate ring of $\mathbb{P}^n$ can be identified with the affine coordinate ring of $\mathbb{C}^{n+1}$, provided that the latter is endowed with the canonical grading $\deg z_i = 1$. It follows that both the Bargmann space of $\mathbb{C}^{n+1}$ and the Hardy space of $\mathbb{P}^n$ are Hilbert space completions of the ring of polynomials in n complex variables, albeit with respect to different scalar products: the former uses the scalar product induced by the flat metric on $\mathbb{C}^{n+1}$, while the latter uses the scalar product induced by the Fubini-Study metric on $\mathbb{P}^n$. According to our general discussion, the relation between the restriction of these scalar products to B_k should be provided by isomorphisms A_k which relate the Berezin quantizations of $\mathbb{C}^{n+1}$ and $\mathbb{P}^n$. In the case at hand, A_k will be proportional to 1_{B_k}, with a k-dependent proportionality constant. This implies that the Berezin quantizations of $\mathbb{P}^n$ and $\mathbb{C}^{n+1}$ agree at any fixed level k. Such an extremely simple relation is not to be expected in general, but is a consequence of the fact that $\mathbb{P}^n$ is a homogeneous space, which is why any reasonable quantization procedure for this space leads to the same result.

Consider an $n + 1$-dimensional complex vector space V and its dual $E = V^*$ as in the previous subsection. As an algebraic variety, the projective space $\mathbb{P}V = (V \setminus \{0\})/\mathbb{C}^*$ over V is given by $\mathrm{Proj}B$, where $B = \mathbb{C}[E]$ is viewed as a graded algebra with respect to the obvious grading. For any vector $v \in V$, we let $[v]$ be the corresponding point in $\mathbb{P}V$. Recall that the tautological bundle $\tau := \mathcal{O}_{\mathbb{P}(V)}(-1)$ has a fiber equal to the line $\tau_v = \mathbb{C}v \subset V$ above the point $[v] \in \mathbb{P}V$. The dual bundle $H := \mathcal{O}_{\mathbb{P}(V)}(1)$ is the hyperplane bundle, which is very ample. Any functional $s \in B_k = (V^*)^{\odot k}$ determines and is determined by a holomorphic section $S \in H^0(H^k)$, namely:

$$S([v]) = s|_{\tau_v^{\otimes k}} \in (\tau_v^*)^{\odot k} \quad ,$$

so $H^0(H^k)$ identifies with B_k. Hence the graded algebra B identifies with the homogeneous coordinate ring $\oplus_{k=0}^\infty H^0(H^k)$ of $\mathbb{P}V$ with respect to H. In particular, $H^0(H)$ identifies with $B_1 = E$.

As in the previous section, a basis $(e_0 \ldots e_n)$ of V determines coordinate functionals $z_j \in E$. The corresponding holomorphic sections $Z_j \in H^0(H)$ are the homogeneous coordinates of $\mathbb{P}V$ associated with the given basis. The homogeneous polynomials (4.12) for $|\mathbf{p}| = k$ provide a basis for B_k. We have:

$$\dim B_k = N_k + 1 = \frac{(n+k)!}{n!k!} \quad .$$

Fixing a scalar product $(\ ,\ )$ on E, we have an induced scalar product on V and take the basis (e_j) to be orthonormal. We endow the hyperplane bundle with the Hermitian

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[14]In particular, the fuzzy two-sphere of [26] is simply the classical Berezin quantization of $\mathbb{P}^1$.

metric h_{FS} specified by:

$$h_{\text{FS}}([v])(Z_j([v]), Z_j([v])) = \frac{|z_j(v)|^2}{|z(v)|^2} = \frac{|v_j|^2}{||v||^2} \quad . \tag{4.16}$$

The associated Kähler metric on $\mathbb{P}V$ is the Fubini-Study metric determined by (,), whose Kähler form is given by:

$$\omega_{\text{FS}}([v]) = \frac{i}{2\pi}\partial\bar{\partial}\log|z(v)|^2 = \frac{i}{2\pi}\partial\bar{\partial}\log||v||^2 \quad .$$

We endow H^k with the tensor product metric $h_{\text{FS}}^k = h_{\text{FS}}^{\otimes k}$, which satisfies:

$$h_{\text{FS}}^k([v])(S([v]), S([v])) = \frac{|s(v)|^2}{||v||^{2k}} \quad \forall s \in B_k \quad .$$

As measure on $\mathbb{P}V$, we use the Liouville measure of the volume form $\frac{\omega_{\text{FS}}^n}{n!}$. In particular, we have:

$$\text{vol}_{\omega_{\text{FS}}}(\mathbb{P}V) = \frac{1}{n!} \quad .$$

The space $H^0(H^k) = B_k$ carries the associated L^2-product:

$$\langle s_1, s_2 \rangle_k := \langle S_1, S_2 \rangle_k^{h_{\text{FS}}^k} = \int_{\mathbb{P}V} \frac{\omega_{\text{FS}}^n}{n!} h_{\text{FS}}^k(S_1, S_2) \quad . \tag{4.17}$$

The monomials $\chi_{\mathbf{p}}$ (such that $|\mathbf{p}| = k$) provide an orthogonal but not orthonormal basis of B_k with respect to this product. Direct computation gives:

$$||\chi_{\mathbf{p}}||_k = \sqrt{\frac{\mathbf{p}!}{(n+k)!}} \quad (|\mathbf{p}| = k) \quad .$$

Comparing with (4.11), we find:

$$\frac{||\chi_{\mathbf{p}}||_k}{||\chi_{\mathbf{p}}||_B} = \frac{1}{\sqrt{(n+k)!}} \quad (|\mathbf{p}| = k) \quad .$$

where $||\chi_{\mathbf{p}}||_B$ is computed with respect to the scalar product of the Bargmann space $\mathcal{B}(\mathbb{C}^{n+1})$. It follows that the Hardy product of the projective Hilbert space $(\mathbb{P}V, \omega_{\text{FS}})$ is related to the Bargmann product by a constant rescaling:

$$\langle s, t \rangle_k = \frac{1}{(n+k)!}\langle s, t \rangle_B \quad \forall s, t \in B_k \quad . \tag{4.18}$$

Let $(\mathcal{E}(\mathbb{P}V), \langle\ ,\ \rangle_{\mathbb{P}V}) = \overline{\oplus}_{k=0}^{\infty}(B_k, \langle\ ,\ \rangle_k)$ denote the Hardy space of $\mathbb{P}V$. It can be identified with the space of those entire functions (4.13) on V whose coefficients satisfy:

$$\langle f|f \rangle_{\mathbb{P}V} = \sum_{\mathbf{p} \in \mathbb{N}^{n+1}} \frac{\mathbf{p}!}{(n+|\mathbf{p}|)!}|a_{\mathbf{p}}|^2 < \infty \quad .$$

Since $(n + \mathbf{p})! \geq 1$, we can view $\mathcal{B}(V)$ as a closed subspace of $\mathcal{E}(\mathbb{P}V)$. (Of course, the Bargmann scalar product does not agree with the restriction of the Hardy product.) The operator $W : \mathcal{E}(\mathbb{P}V) \to \mathcal{B}(V) \subset \mathcal{E}(\mathbb{P}V)$ given by:

$$(Wf)(z) = \sum_{\mathbf{p} \in \mathbb{N}^{n+1}} \sqrt{(n + |\mathbf{p}|)!} a_{\mathbf{p}} z^{\mathbf{p}} \qquad (4.19)$$

provides an isometry between $(\mathcal{E}(\mathbb{P}V), \langle \ , \ \rangle_{\mathbb{P}V})$ and $(\mathcal{B}(V), \langle \ , \ \rangle_B)$. Defining $A := W^2$, we have $A_k := A|_{B_k} = (n + |\mathbf{p}|)! 1_{B_k}$ and:

$$\langle s, t \rangle_{\mathbb{P}V} = \langle As, t \rangle_B \ .$$

Since $\mathbb{P}V$ is a homogeneous space, its coherent states can be extracted by the well-known method due to Perelomov [27]. The unitary group $\mathrm{U}(V)$ of the Hermitian vector space $(V, (\ , \ ))$ acts unitarily on B_k via $\hat{U}_k(f)(v) = f(U^{-1}v)$, i.e. $\hat{U}_k|v, k\rangle = |Uv, k\rangle$. The representation is irreducible and isomorphic with the k-fold symmetric representation. One can construct Perelomov coherent states [27] of this action as orbits of a given non-vanishing state in the projective Hilbert space $\mathbb{P}(B_k)$. Let us start with the state defined by the vector $|v_0, k\rangle \in B_k$, where v_0 is any fixed non-vanishing vector of V. Then the ray $\mathbb{C}^*|v_0, k\rangle \in B_k$ has stabilizer $\mathrm{U}(v_0^\perp) \times \mathrm{U}(1)$ in $\mathrm{U}(V)$. It follows that the Perelomov states are parameterized by points of the homogeneous space $\mathrm{U}(V)/(\mathrm{U}(v_0^\perp) \times \mathrm{U}(1))$, which coincides with $\mathbb{P}V$. Since $\hat{U}_k|z_0, k\rangle = |Uz_0, k\rangle$, the Perelomov state at $[z]$ coincides with the ray $\mathbb{C}^*|z, k\rangle$. Hence Perelomov's coherent projectors take the form:

$$P_{[v]}^{(k)} := \frac{|v, k\rangle\langle v, k|_B}{\langle v, k|v, k\rangle_B} = k! \frac{|v, k\rangle\langle v, k|_B}{||v||^{2k}} \ . \qquad (4.20)$$

These projectors are unaffected by the constant rescaling of the scalar product when translating between the Bargmann and Hardy metrics on B_k. Since the Fubini-Study metric is invariant on the homogeneous space $\mathbb{P}V$, the Liouville form determined by its volume form $\frac{\omega_{\mathrm{FS}}^n}{n!}$ provides an invariant measure. Thus Perelomov's theory implies the overcompleteness property:

$$\frac{N_k + 1}{\mathrm{vol}(\mathbb{P}V)} \int_{\mathbb{P}V} \frac{\omega_{\mathrm{FS}}^n}{n!} P_{[v]}^{(k)} = P_k \ , \qquad (4.21)$$

where P_k is the orthoprojector on B_k in $\mathcal{B}(V)$ and the normalization constant in front of the integral has been identified by taking the trace. Since both Rawnsley's and Perelomov's coherent projectors satisfy (4.21), they determine a reproducing kernel for B_k, so they must agree with each other if one uses scalar products on this space differing only by a constant rescaling. It follows that the $P_{[v]}^{(k)}$ coincide with the Rawnsley projectors of $(B_k, \langle \ , \ \rangle_k)$ while the rays $\mathbb{C}^*|v, k\rangle$ are Rawnsley's coherent states. Rawnsley's coherent vectors take the form:

$$e_v^{(k)} = (n + k)! |v, k\rangle \quad (v \in E \setminus \{0\}) \ , \qquad (4.22)$$

where the prefactor is due to relation (4.18) between the Hardy and Bargmann scalar products. The reproducing kernel for $(B_k, \langle \ , \ \rangle_k)$ is given by:

$$K_k(u, v) = \langle e_u^{(k)}|e_v^{(k)}\rangle_k = (n + k)! \langle u, k|v, k\rangle_B = \frac{(n + k)!}{k!} (u \cdot \bar{v})^k \ .$$

The epsilon function is constant:

$$\epsilon_k^{\mathbb{P}V} = \frac{N_k + 1}{\text{vol}(\mathbb{P}V)} = \frac{(n+k)!}{k!} \quad . \tag{4.23}$$

In particular, we recover the well-known fact that $k\omega_{\text{FS}}$ is balanced for all k. The over-completeness property (4.21) can also be written as:

$$(n+k)! \int_X \frac{\omega_{\text{FS}}^n}{n!} \frac{|v,k\rangle\langle v,k|_B}{||v||^{2k}} = P_k \quad ,$$

where we used $\text{vol}(\mathbb{P}V) = \frac{1}{n!}$ and the identity $k!(N_k + 1) = \frac{(n+k)!}{n!}$.

It will prove convenient to consider the functions:

$$f_{IJ} := \frac{\bar{z}^I z^J}{|z|^{2m}} = \frac{\bar{z}_0^{i_0} \dots \bar{z}_n^{i_n} z_0^{j_0} \dots z_n^{j_n}}{|z|^{2m}} \in \mathcal{C}^\infty(\mathbb{P}V, \mathbb{C}) \quad , \tag{4.24}$$

where $I = (i_0 \dots i_n), J = (j_0 \dots j_n) \in \mathbb{N}^{n+1}$ and $|I| = |J| = m$ and where we set $f_{IJ} = 1$ for $m = 0$. The linear span $\mathcal{S}(\mathbb{P}V)$ over $\mathbb{C}$ of these functions is a unital $*$-subalgebra of the C^*-algebra $(\mathcal{C}^\infty(\mathbb{P}V), || \ ||_\infty)$ (recall that $|| \ ||_\infty$ is the sup norm). Let $\mathcal{S}_m(\mathbb{P}V)$ be the subspace spanned by those f_{IJ} with $|I| = |J| = m$ (notice that $\mathcal{S}_0 = \mathbb{C}$). The set of functions (4.24) with a fixed m gives a basis for $\mathcal{S}_m(\mathbb{P}V)$ so in particular we have $\dim_\mathbb{C} \mathcal{S}_m(\mathbb{P}V) = N_m + 1$. For any $l = 0 \dots n$, let $\Delta_l \in \mathbb{N}^{n+1}$ be given by $\Delta_l(i) = \delta_{il}$. The obvious relation:

$$f_{IJ} = \sum_{l=0}^n f_{I+\Delta_l, J+\Delta_l}$$

shows that $\mathcal{S}_m(\mathbb{P}V) \subset \mathcal{S}_{m+1}(\mathbb{P}V)$ for all $m \geq 0$, so that $\mathcal{S}(\mathbb{P}V) = \cup_{m=0}^\infty \mathcal{S}_m(\mathbb{P}V)$ is a filtered $*$-algebra. Notice that $\mathcal{S}(\mathbb{P}V)$ is generated as a $*$-algebra by the elements $f_{ij} = \frac{\bar{z}_i z_j}{|z|^2} \in \mathcal{S}_1$.

Proposition. The $*$-subalgebra $\mathcal{S}(\mathbb{P}V)$ is dense in the C^*-algebra $(\mathcal{C}^\infty(\mathbb{P}V), || \ ||_\infty)$.

Proof. Given a point $[v] \in \mathbb{P}V$, there exists an index $i = 0 \dots n$ such that $z_i(v) \neq 0$. In particular $f_{ii}([v]) = \frac{|z_i(v)|^2}{|z(v)|^2} \neq 0$. It follows that $\mathcal{S}_1$ separates points. Since $\mathcal{S}_1$ generates $\mathcal{S}$ as a $*$-algebra, the conclusion follows from the Stone-Weierstraß theorem.

Toeplitz quantization of $\mathbb{P}V$. The Toeplitz quantization map $T_k : \mathcal{C}^\infty(\mathbb{P}V, \mathbb{C}) \to \text{End}(B_k)$ takes the form:

$$T_k(f) = \frac{N_k + 1}{\text{vol}(\mathbb{P}V)} \int_{\mathbb{P}V} \frac{\omega_{\text{FS}}^n}{n!} f(x) P_x^{(k)} = (n+k)! \int_{\mathbb{P}V} \frac{\omega_{\text{FS}}^n}{n!} f([v]) \frac{|v,k\rangle\langle v,k|_B}{||v||^{2k}} \quad .$$

This map is surjective since $\mathbb{P}V$ is compact. Using relations (4.15), we find:

$$T_k(f_{IJ}) = \frac{(n+k)!}{n!} \int_{\mathbb{P}V} \omega_{\text{FS}}^n \frac{\hat{a}^I |v,k+d\rangle\langle v,k+d|_B (\hat{a}^\dagger)^J}{|z|^{2(k+m)}} = \frac{(n+k)!}{(n+k+m)!} \hat{a}^I P_{k+m} (\hat{a}^\dagger)^J \quad .$$

In particular, we have:

$$T_k(f_{ij}) = \frac{1}{n+k+1} \hat{a}_i \hat{a}_j^\dagger \quad .$$

Berezin quantization of $\mathbb{P}V$. The Berezin symbol map $\sigma_k : \text{End}(B_k) \to \mathcal{C}^\infty(\mathbb{P}V, \mathbb{C})$ takes the form:

$$\sigma_k(C)([v]) = \frac{\langle v, k|C|v, k\rangle}{\langle v, k|v, k\rangle} \quad \forall C \in \text{End}(B_k) \ .$$

This map is injective and its inverse on the image $\Sigma_k(\mathbb{P}V) := \text{im}\,\sigma_k$ defines the Berezin quantization $Q_k : \Sigma_k(\mathbb{P}V) \to \text{End}(B_k)$, which is a linear isomorphism. For the functions (4.24), we find:

$$Q_k(f_{IJ}) = \frac{(k-m)!}{k!} P_k(\hat{a}^\dagger)^I \hat{a}^J P_k$$

and in particular:

$$Q_k(f_{ij}) = \frac{1}{k} \hat{a}_j^\dagger \hat{a}_i \ .$$

Notice that $Q_k(f_{IJ})$ vanishes for $m \geq k$. Since the operators $\hat{f}_{IJ} = P_k(\hat{a}^\dagger)^I \hat{a}^J P_k$ with $m := |I| = |J| = k$ provide a basis for $\text{End}(B_k)$, it follows that the image $\Sigma_k(\mathbb{P}V)$ of the Berezin symbol map coincides with $\mathcal{S}_k(\mathbb{P}V)$:

$$\Sigma_k(\mathbb{P}V) = \mathcal{S}_k(\mathbb{P}V) \ \ \forall k \geq 1 \ .$$

It follows that $\Sigma_k(\mathbb{P}V)$ provides a weakly exhaustive filtration of $(\mathcal{C}^\infty(\mathbb{P}V), ||\ ||_\infty)$:

$$\overline{\cup_{k=1}^\infty \Sigma_k(\mathbb{P}V)} = \mathcal{C}^\infty(\mathbb{P}V) \ .$$

The Berezin transform $\beta_k : \mathcal{C}^\infty(\mathbb{P}V) \to \Sigma_k(\mathbb{P}V)$ takes the form:

$$\beta_k(f)(v) = \sigma_k(T_k(f)) = \frac{(n+k)!}{k!} \int_{\mathbb{P}V} \frac{\omega_{\text{FS}}^n}{n!}(u) \left(\frac{|(u,v)|}{||u||||v||}\right)^{2k} \ .$$

Notice that again Berezin and Toeplitz quantizations use Wick and anti-Wick orderings, respectively. An extension of this quantization providing access to vector bundles over quantized $\mathbb{P}^n$ has been presented in [30].

5. Berezin-Bergman quantization

In this section we discuss a generalized Berezin quantization procedure which clarifies the proposal of [1]. This prescription, which we call Berezin-Bergman quantization, is relevant for compact complex manifolds endowed with a Bergman metric.

Let (X, L) be a polarized compact complex manifold and assume that L is very ample with $\dim_\mathbb{C} H^0(L) = n + 1$. We let $E_k := H^0(L^k)$ and $\dim_\mathbb{C} E_k = M_k + 1$ (thus $M_1 = n$). The homogeneous coordinate ring $R(X, L) = \oplus_{k=0}^\infty H^0(L^k) = \oplus_{k=0}^\infty E_k$ of X with respect to L is generated in degree one and we have an isomorphism of graded algebras:

$$\phi : R \xrightarrow{\sim} B/I \ , \tag{5.1}$$

where $B = \oplus_{k=0}^\infty E_1^{\odot k}$ is the symmetric algebra over the vector space $E_1 := H^0(L)$ and I is a graded ideal in B generated in degrees ≥ 2. The algebra B can be identified with the algebra of polynomial functions on E_1^*, and thus with the coordinate ring $\mathbb{C}[E_1^*]$ of

the affine space $\mathrm{Spec}B$ over $E_1^* \simeq \mathbb{C}^{n+1}$. As a graded algebra, it is also the homogeneous coordinate ring of the projective space $\mathbb{P}[E_1^*]$. The Kodaira embedding $i : X \hookrightarrow \mathbb{P}(E_1^*)$ defined by L presents X as a projective variety in $\mathbb{P}[E_1^*]$, whose vanishing ideal equals I, and whose homogeneous coordinate ring equals R. Writing $B = \oplus_{k=0}^{\infty} B_k$ and $I = \oplus_{k=0}^{\infty} I_k$, the homogeneous components satisfy $I_k \subset B_k$ as well as:

$$E_k \simeq B_k/I_k \ .$$

Let us now consider a scalar product $(\, , \,)_1$ on E_1 and the associated Bergman metric on X, whose Kähler form we denote by ω. We also let h be the induced Bergman Hermitian scalar product on L. For every $k \geq 1$, we have *two* natural ways to induce a scalar product on $H^0(L^k)$. The first choice is to take the L^2-product:

$$\langle s, t \rangle_k = \int_X \frac{\omega^n}{n!} h_k(s,t) \ ,$$

where $h_k = h^{\otimes k}$. Performing generalized Berezin quantization with respect to this sequence of products leads to the classical Berezin-Toeplitz theory discussed in the previous section.

The second choice is as follows. The product $(\, , \,)_1$ on E_1 induces a scalar product $(\, , \,)_B$ on the symmetric algebra $B = \oplus_{k=0}^{\infty} E_1^{\odot k}$ via the prescription:

$$(s_1 \odot \ldots s_k, t_1 \odot \ldots t_l)_B = \frac{1}{k!} \delta_{k,l} \sum_{\sigma \in S_k} (s_1, t_{\sigma(1)})_1 \ldots (s_k, t_{\sigma(k)})_1 \ , \tag{5.2}$$

where S_k is the symmetric group on k letters and $s_i, t_i \in E_1$. Notice that the completion of $B = \oplus_{k=0}^{\infty} E_1^{\odot k}$ with respect to the product (5.2) is the bosonic Fock space over the $n+1$-dimensional Hilbert space $(E_1, (\, , \,)_1)$. Of course, this can also be viewed as the Bargmann space over $V = E_1^*$, which appeared in the quantization of the affine space $A[V]$. Thus we can view B as embedded in the Bargmann space $\mathcal{B}(V)$, and (5.2) is the restriction of the Bargmann product (4.10) to B.

Using the scalar product (5.2), we can identify $E_k \simeq B_k/I_k$ with the orthogonal complement $I_k^{\perp} = \{s \in B_k | (s,t)_B = 0 \ \ \forall t \in I_k\}$ of I_k in B_k. This identification gives a scalar product $(\, , \,)_k$ on E_k, which is induced by the restriction of $(\, , \,)_B$ to $I_k^{\perp}$. To state this precisely, notice that the Kodaira embedding $i : X \hookrightarrow \mathbb{P}[E_1^*]$ defined by L allows us to identify B_k with the space of holomorphic sections of H^k, where H is the hyperplane bundle $H = \mathcal{O}_{\mathbb{P}[E_1^*]}(1)$:

$$B_k = H^0(H^k) \ .$$

Furthermore, the homogeneous component I_k of the vanishing ideal I can be identified with the kernel of the pull-back map (restriction) on sections $i_k^* : H^0(H^k) = B_k \to H^0(L^k) = E_k$:

$$I_k := \ker i_k^* \ .$$

Since i_k^* is surjective, it induces an isomorphism $\psi_k : I_k^{\perp} \to E_k$, whose inverse $\phi_k := \psi_k^{-1} : E_k \to I_k^{\perp} \simeq B_k/I_k$ we can take as the homogeneous k-component of (5.1). We define $(\, , \,)_k$ as follows:

$$(s, t)_k := \alpha_k (\phi_k(s), \phi_k(t))_B \ , \tag{5.3}$$

where the scaling constants:

$$\alpha_k := \frac{\mathrm{vol}_\omega(X)}{\mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}V)} \frac{N_k + 1}{M_k + 1}$$

are chosen for later convenience. Here, $\mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}V) = \frac{1}{n!}$.

Definition. The *Berezin-Bergman quantization* of (X, L) determined by the scalar product $(\ ,\ )_1$ on $H^0(L)$ is the generalized Berezin quantization performed with respect to the sequence of scalar products $(\ ,\ )_k$ on $H^0(L^k)$ defined in (5.3).

Using the orthogonal decomposition $B_k = I_k \oplus I_k^\perp$, let us pick a $(\ ,\ )_B$-orthonormal basis $S_0 \ldots S_{N_k}$ of B_k such that $S_0 \ldots S_{M_k}$ is a basis of $I_k^\perp$ and such that $S_{M_k+1} \ldots S_{N_k}$ is a basis of I_k. Then $i_k^*(S_j) = 0$ for $j > M_k$ and the sections $s_j := \frac{1}{\sqrt{\alpha_k}} i_k^*(S_j)$ (with $j = 0 \ldots M_k$) give an orthonormal basis of the space $(E_k, (\ ,\ )_k)$. The epsilon function of $\mathbb{P}V$ at level k takes the form:

$$\epsilon_k^{\mathbb{P}V}([v]) = \sum_{j=0}^{N_k} h_{\mathrm{FS}}^k([v])(S_j([v]), S_j([v])) = \frac{N_k + 1}{\mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}V)} \quad .$$

Restricting this identity to X shows that the epsilon function of the pair $(h_k, (\ ,\ )_k)$ is constant on X:

$$\epsilon_k(x) = \sum_{j=0}^{M_k} h_k(s_j(x), s_j(x)) = \frac{M_k + 1}{\mathrm{vol}_\omega(X)} \quad (x \in X) \quad .$$

In particular, the induced scalar product $(\ ,\ )_k$ coincides with the L^2-scalar product $\langle\ ,\ \rangle_k$ iff the epsilon function of the latter is constant, i.e. iff $k\omega$ is balanced.

We now consider the generalized Berezin quantization of (X, ω, L, h) with respect to the sequence of induced scalar products $(\ ,\ )_k$ on $E_k = H^0(L^k), k \geq 1$. We have surjective Berezin symbol maps $\sigma_k : \mathrm{End}(E_k) \to \mathcal{C}^\infty(X)$ whose images we denote by Σ_k, and associated quantization maps $Q_k := (\sigma_k|_{\Sigma_k})^{-1} : \Sigma_k \to \mathrm{End}(E_k)$.

Let Λ_k be the orthogonal projector of B_k onto $I_k^\perp$ with respect to the product (5.2). It is easy to see that the Rawnsley coherent states of E_k with respect to $(\ ,\ )_k$ are given by:

$$e_v^{(k)} := \frac{1}{\alpha_k} i_k^*(\Lambda_k|v, k\rangle) = \frac{1}{\alpha_k} i^*(|v, k\rangle) \quad \forall v \in C(X) \setminus \{0\} \subset V \quad ,$$

where $C(X) = \mathrm{Spec}(B/I) \subset V$ is the affine cone over X. The second form follows from the fact that the component $(1 - \Lambda_k)|z, k\rangle$ along I_k vanishes for $z \in C(X)$, so that:

$$\Lambda_k|v, k\rangle = |v, k\rangle \quad \text{for} \quad v \in C(X) \quad .$$

The Rawnsley projectors take the form:

$$P_{[v]}^{(k)} = i_k^* \circ \frac{|v, k\rangle\langle v, k|_B}{\langle v, k|v, k\rangle_B} \circ \phi_k \quad ([v] \in X \subset \mathbb{P}V) \quad , \tag{5.4}$$

while the Berezin symbol of an operator $C \in \operatorname{End}(E_k)$ is given by:

$$\sigma_k(C)([v]) = \frac{\langle v, k | \tilde{C} | v, k \rangle_B}{\langle v, k | v, k \rangle_B} \quad , \tag{5.5}$$

where $\tilde{C} := \phi_k \circ C \circ i_k^* = \phi_k \circ C \circ \phi_k^{-1} \circ \Lambda_k \in \operatorname{End}(B_k)$ is the transport of C to an operator on B_k through the isomorphism $\phi_k : E_k \to I_k^\perp \subset B_k$.

Recall that the operators $\hat{f}_{IJ} := P_k(\hat{a}^\dagger)^I \hat{a}^J P_k$ (where $|I| = |J| = k$) form a basis of $\operatorname{End}(B_k)$. Thus space $\operatorname{End}(I_k^\perp)$ is spanned by the operators $\Lambda_k \hat{f}_{IJ} \Lambda_k$ and $\operatorname{End}(E_k)$ is spanned by:

$$\hat{f}'_{IJ} := i_k^* \circ \hat{f}_{IJ} \circ \phi_k \quad (|I| = |J| = k) \quad .$$

We have $\tilde{\hat{f}}'_{IJ} := \Lambda_k \hat{f}_{IJ} \Lambda_k$. Applying (5.5) to these operators, we find:

$$\sigma_k(\hat{f}'_{IJ})([v]) = \frac{\langle v, k | \hat{f}_{IJ} | v, k \rangle_B}{\langle v, k | v, k \rangle_B} = f_{IJ}([v]) \qquad ([v] \in X) \quad .$$

It follows that:

$$Q_k(f_{IJ}|_X) = \hat{f}'_{IJ} \quad ,$$

where $f_{IJ}|_X := f_{IJ} \circ i \in \mathcal{C}^\infty(X)$. We conclude that $\Sigma_k(X)$ is spanned by the restrictions $f_{IJ}|_X$ with $|I| = |J| = k$. Since $\mathcal{S}_k(\mathbb{P}V) \subset \mathcal{S}_{k+1}(\mathbb{P}V)$, we have $\Sigma_k(X) \subset \Sigma_{k+1}(X)$. The union $\mathcal{S}(X) := \cup_{k \geq 0} \Sigma_k(X)$ (where $\Sigma_0 := \mathbb{C}$ consists of the constant functions on X) is a filtered unital $*$-subalgebra of the C^*-algebra $(\mathcal{C}^\infty(X), \| \|_\infty)$. Since the $*$-algebra $\mathcal{S}(\mathbb{P}V)$ is generated by $\mathcal{S}_1(\mathbb{P}V)$, it follows that $\mathcal{S}(X)$ is generated by $\Sigma_1(X)$ as a $*$-algebra. Moreover, $\Sigma_1(X)$ separates the points of X since X can be viewed as a subset of $\mathbb{P}V$ and since $\mathcal{S}_1(X)$ separates the points of the latter. It follows that $\mathcal{S}(X)$ is dense in $(\mathcal{C}^\infty(X), \| \|_\infty)$.

Remark. Let I be generated by p homogeneous polynomials $F_1 \ldots F_p$ of degrees at least two. Since $\hat{a}_i$ act on the Bargmann space as multiplication by z_i, we have the linear decomposition $I = \sum_{l=1}^p \operatorname{im} F_l(\hat{a})$, where im denotes the image of a linear operator and all operators are taken to act in the space B. It follows that:

$$I^\perp = \cap_{l=1}^p \ker \bar{F}_l(\hat{a}^\dagger) \quad ,$$

where $\bar{f}$ is the polynomial in $z_0 \ldots z_n$ obtained by conjugating all *coefficients* of f. Since $\hat{a}_i^\dagger$ act as ∂_i, the operators in the right hand side are holomorphic differential operators:

$$\bar{F}_l(\hat{a}^\dagger) = \bar{F}_l(\partial_0 \ldots \partial_n)$$

and we find that $I^\perp$ is the graded vector space of those solutions s of the system of the following linear holomorphic partial differential equations with constant coefficients:

$$\bar{F}_l(\partial_0 \ldots \partial_n) s(v_0 \ldots v_n) = 0 \quad , \tag{5.6}$$

which are homogeneous polynomials in $v_0 \ldots v_n$. The graded components $I_k^\perp \subset B_k$ can be obtained by restricting to homogeneous polynomials of degree k, which amounts to imposing the condition:

$$Gs = ks \quad , \tag{5.7}$$

where $G = \sum_{i=0}^{n} v^i \partial_i$ is the Euler operator. These equations determine the unique extension of a section $s \in H^0(X, L^k)$ to a section of $H^0(\mathbb{P}V, H^k)$ lying in $I_k^\perp$. The point is that the system (5.6), (5.7) has a unique polynomial solution which has prescribed behavior on the affine cone $C(X) \subset V$.

Relation with fuzzy geometry. A procedure for defining fuzzy versions of compact Hodge manifolds X was proposed in [1]. It is clear from the above that:

The proposal of [1] amounts to defining the fuzzy version of X as the generalized Berezin quantization of X with respect to the sequence of induced scalar products $(\ ,\ )_k$ on the spaces $E_k := H^0(L^k)$. This quantization prescription agrees with classical Berezin quantization at a fixed level k iff $k\omega$ is balanced. It is also easy to prove the following:

Proposition. The Berezin-Bergman quantization of $\mathbb{P}V$ coincides with its classical Berezin quantization.

Proof. The vanishing ideal I is zero in this case, so the Berezin-Bergman scalar product $(\ ,\ )_k$ of section 5 coincides with the restriction of the Bargmann product $\langle\ ,\ \rangle_B$ (4.10) to E_k (recall (5.2) the intermediate scalar product $(\ ,\ )_B$ induced by $(\ ,\ )_1$). Since $\langle\ ,\ \rangle_B$ coincide with the L^2-product $\langle\ ,\ \rangle_k$ up to a constant scale factor, the results of section 3 show that the generalized Berezin quantization based on $(\ ,\ )_k$ is equivalent to that based on $\langle\ ,\ \rangle_k$. The first quantization is the Berezin-Bergman quantization, while the second is the classical Berezin quantization of $\mathbb{P}V$.

6. Harmonic analysis on quantized Hodge manifolds

In this section, we discuss the construction of a "fuzzy" Laplace operator on Berezin quantized compact Hodge manifolds. There are two obvious choices: the Berezin push and the Berezin-Toeplitz lift of the ordinary Laplace operator. We use the former to calculate the approximate spectrum of the Laplace operator for two examples and employ the latter to define fuzzy scalar field theories on arbitrary compact Hodge manifolds. Let us fix a compact prequantized Hodge manifold (X, ω, L, h).

6.1 'Quantizing' the classical laplacian

Let us take μ to be the Liouville measure defined by the Kähler form ω and fix some generalized Berezin quantization of (X, ω). Recall (3.26) that the space $\mathcal{C}^\infty(X)$ carries the L^2-scalar product induced by the Kähler metric:

$$\prec f, g \succ = \int_X \frac{\omega^n}{n!} \bar{f} g \tag{6.1}$$

as well as the scalar products (3.27):

$$\prec f, g \succ_{\epsilon_k} = \int_X \frac{\omega^n}{n!} \epsilon_k \bar{f} g \ ,$$

On the other hand, the Berezin algebras $(\Sigma_k, \diamond_k)$ (where $\Sigma_k = \sigma(\mathrm{End}(E_k))$ with $E_k = H^0(L^k)$) carry the Berezin scalar products (3.29) induced by the trace (3.28):

$$\prec f, g \succ_B = \int_k \bar{f} \diamond_k g = \int_X \frac{\omega^n}{n!} \epsilon_k f \diamond_k g = \langle Q_k(f), Q_k(g) \rangle_{\mathrm{HS}} \quad . \tag{6.2}$$

The Laplace operator Δ of (X, ω) is Hermitian and positive with respect to the ordinary L^2-scalar product (6.1) on functions, which is the realization of (3.26) in the case at hand.

The Berezin push. Considering the symbol spaces Σ_k, we define truncated Laplacians $\Delta_k : \Sigma_k \to \Sigma_k$ via:

$$\Delta_k = \pi_k \circ \Delta|_{\Sigma_k} \quad ,$$

where π_k is the $\prec, \ \succ$-orthoprojector on Σ_k inside $\mathcal{C}^\infty(X)$. Explicitly, let the Kähler form be given by $\omega = \omega_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ in some local coordinates z^i defined on a Zariski open set and let Σ_k be spanned by the functions $e_i \in \mathcal{C}^\infty(X)$ with $i = 0, \ldots, N_k$. The Laplace operator of (X, ω) takes the form:

$$\Delta f \ = \ \omega^{i\bar{j}} \partial_i \partial_{\bar{j}} f \quad ,$$

and the orthoprojector $\pi_k : \mathcal{C}^\infty(X) \to \Sigma_k$ is given by the map:

$$\pi_k(f) = \sum_{i=0}^{N_k} e_i \int_X \frac{\omega^n}{n!} \bar{e}_i f \quad (f \in \mathcal{C}^\infty(X)) \quad . \tag{6.3}$$

Notice that Δ_k is $\prec, \ \succ$-self-adjoint and positive on Σ_k. The truncated Laplacian need not be Hermitian with respect to the Berezin scalar product, so the Berezin push $\Delta_k^B = Q_k \circ \Delta_k \circ \sigma_k$ may generally fail to be Hermitian with respect to the Hilbert-Schmidt scalar product on $\mathrm{End}(E_k)$. It follows that the Berezin push Δ_k^B does *not* provide a good general notion of "fuzzy Laplacian". In fuzzy field theory, the fuzzy Laplacian is used when building the kinetic term for scalar fields in the "fuzzified" field action, which is defined on $\mathrm{End}(E_k)$. Since the natural scalar product on that space is the Hilbert-Schmidt product, the kinetic term should be specified by an operator which is $\langle, \ \rangle_{\mathrm{HS}}$-Hermitian and positive.

The Berezin-Toeplitz lift. The discussion of section 3.8 shows that the Berezin-Toeplitz lift (3.34) of the Laplacian:

$$\hat{\Delta}_k := T_k \circ M_{\frac{1}{\epsilon_k}} \circ \Delta \circ \sigma_k : \mathrm{End}(E_k) \to \mathrm{End}(E_k)$$

is a positive Hermitian operator on $(\mathrm{End}(E_k), \langle, \ \rangle_{\mathrm{HS}})$. Moreover, the Berezin-Toeplitz transform (3.37):

$$\Delta_{\diamond_k} = \beta_{mod,k} \circ \Delta|_{\Sigma_k} = \beta_k \circ M_{\frac{1}{\epsilon_k}} \circ \Delta|_{\Sigma_k} : \Sigma_k \to \Sigma_k$$

is Hermitian and positive-definite with respect to the Berezin product (6.2). We will view $\hat{\Delta}_k$ (equivalently, $\Delta_{\diamond_k}$) as the definition of the "fuzzy Laplacian" at level k. Explicitly, we have:

$$\hat{\Delta}_k(C) = \int_X \frac{\omega^n}{n!}(x) P_x^{(k)}(\Delta \sigma_k(C))(x) \quad (C \in \mathrm{End}(E_k))$$

and:

$$\Delta_{\diamond_k}(f)(x) = \int_X \frac{\omega^n}{n!}(y)\Psi_k(x,y)(\Delta f)(y) \quad (f \in \Sigma_k) \ ,$$

where we used the integral expression (3.36) for the modified Berezin transform:

$$\beta_{mod,k}(f)(x) := \int_X \frac{\omega^n}{n!}(y)\Psi_k(x,y)f(y) \ .$$

We are using Ψ_k which is the squared two-point function (3.10) at level k:

$$\Psi_k(x,y) := \sigma_k(P_x^{(k)})(y) = \sigma_k(P_y^{(k)})(x) = \mathrm{tr}\left(P_x^{(k)}P_y^{(k)}\right) \ .$$

6.2 The case of Kähler homogeneous spaces

Recall that a Kähler manifold (X,ω) is called a Kähler homogeneous space when its group of holomorphic isometries $\mathrm{Aut}(X,\omega)$ acts transitively on X. In this case, the equivariance properties of Berezin and Berezin-Toeplitz quantization allow one to use representation-theoretic arguments in order to extract further information. Let us consider the case of simply connected coadjoint orbits $X = G/H$, where we can take G to be a compact simple Lie group. This includes the case of projective spaces $\mathbb{P}^n$, which corresponds to the choice $G = \mathrm{SU}(n+1)$. In this situation $G = \mathrm{Aut}_{L,h}(X,\omega) = \mathrm{Aut}(X,\omega)$. This class of spaces has been studied in great detail, so we will only make a few basic remarks for use in the next subsection.

Consider the classical Berezin and Berezin-Toeplitz quantizations of such a space. The general discussion of section 3 shows that all objects associated with these quantization schemes are G-equivariant. In particular, G-invariance of the (absolute) epsilon function implies that it is constant and given by:

$$\epsilon_k = \frac{N_k + 1}{\mathrm{vol}_\omega(X)} \ ,$$

where $N_k + 1 = \dim E_k$. The unitary representation ρ_k of G in E_k gives the decomposition:

$$E_k = \oplus_{\theta \in \mathrm{Rep}(G)} R_\theta \otimes_{\mathbb{C}} E_k(\theta) \ ,$$

where $\mathrm{Rep}(G)$ is the discrete set of unitary finite-dimensional irreps of the compact Lie group G, R_θ are the corresponding G-modules and $E_k(\theta)$ are (possibly zero) Hermitian vector spaces. Their dimensions $m_k(\theta) := \dim E_k(\theta) \geq 0$ are the corresponding multiplicities. Of course, only a finite number of irreps appear with non-zero multiplicity in the sum above, i.e. we have $E_k(\theta) = 0$ except for a finite number of values of θ. The Hermitian vector space $(\mathrm{End}(E_k), \langle, \rangle_{\mathrm{HS}}) = E_k^* \otimes E_k$ carries the unitary representation $\hat{\rho}_k = \rho_k^* \otimes \rho_k$, whose orthogonal decomposition into irreducibles we write as:

$$\mathrm{End}(E_k) = \oplus_{\theta \in \mathrm{Rep}(G)} R_\theta \otimes_{\mathbb{C}} W_k(\theta) \ . \tag{6.4}$$

The equivariance property (3.14) of the bijection $Q_k : \Sigma_k \to \mathrm{End}(E_k)$ shows that the Berezin quantization map is an isomorphism between the unitary G-modules $(\Sigma_k, \prec, \succ_B$

$,\tau|_{\Sigma_k})$ and $(\mathrm{End}(E_k),\langle\,,\,\rangle_{\mathrm{HS}},\hat{\rho}_k)$. Accordingly, Σ_k consists of representation functions for G and has the $\prec\,,\;\succ_B$-orthogonal decomposition:

$$\Sigma_k = \oplus_{\theta\in\mathrm{Rep}(G)}R_\theta\otimes_\mathbb{C}\Sigma_k(\theta)\ ,\tag{6.5}$$

while Q_k has the form:

$$Q_k = \oplus_{\theta\in\mathrm{Rep}(G)}\mathrm{id}_{R_\theta}\otimes Q_k(\theta)\tag{6.6}$$

for some isometries $Q_k(\theta):\Sigma_k(\theta)\to W_k(\theta)$. The Berezin symbol map is also τ-equivariant and thus takes the form:

$$\sigma_k = \oplus_{\theta\in\mathrm{Rep}(G)}\mathrm{id}_{R_\theta}\otimes\sigma_k(\theta)\ ,\tag{6.7}$$

where $\sigma_k(\theta)=Q_k(\theta)^{-1}:W_k(\theta)\to\Sigma_k(\theta)$. A similar argument shows that the (restricted) Toeplitz quantization map takes the form:

$$T_k|_{\Sigma_k} = \oplus_{\theta\in\mathrm{Rep}(G)}\mathrm{id}_{R_\theta}\otimes T_k(\theta)\tag{6.8}$$

for some bijections $T_k(\theta):\Sigma_k(\theta)\to W_k(\theta)$. Unlike Q_k, the operators $T_k(\theta)$ need not be not unitary since T_k need not be unitary with respect to the scalar products $\prec\,,\;\succ_B$ and $\langle\,,\,\rangle_{\mathrm{HS}}$. Combining the above, we find that the (restricted) Berezin transform decomposes as:

$$\beta|_{\Sigma_k} = \oplus_{\theta\in\mathrm{Rep}(G)}\mathrm{id}_{R_\theta}\otimes\beta_k(\theta)\ ,\tag{6.9}$$

where $\beta_k(\theta)$ are linear automorphisms of the subspaces $\Sigma_k(\theta)$.

Since $X=G/H$ is a Kähler homogeneous space, its Laplace operator $\Delta:\mathcal{C}^\infty(X)\to\mathcal{C}^\infty(X)$ is G-invariant:

$$\Delta\circ\tau = \tau\circ\Delta \quad (\tau\in\mathrm{Aut}(X,\omega))\ .$$

It follows that[15] $\Delta(\Sigma_k)\subset\Sigma_k$ (thus $\Delta_k=\Delta|_{\Sigma_k}$) and that we have a decomposition:

$$\Delta_k = \oplus_{\theta\in\mathrm{Rep}(G)}\mathrm{id}_{R_\theta}\otimes\Delta_k(\theta)\tag{6.10}$$

for some linear operators $\Delta_k(\theta)$ acting in the spaces $\Sigma_k(\theta)$. It is now clear that the Berezin push (3.7) and the Berezin-Toeplitz lift (3.34) of Δ take the forms:

$$\Delta_k^B = \oplus_{\theta\in\mathrm{Rep}(G)}\mathrm{id}_{R_\theta}\otimes\Delta_k^B(\theta)$$

and:

$$\frac{N_k+1}{\mathrm{vol}_\omega(X)}\hat{\Delta}_k = \oplus_{\theta\in\mathrm{Rep}(G)}\mathrm{id}_{R_\theta}\otimes\hat{\Delta}_k(\theta)\ ,$$

where $\Delta_k^B(\theta)=Q_k(\theta)\circ\Delta_k(\theta)\circ\sigma_k(\theta)$ and $\hat{\Delta}_k(\theta)=T_k(\theta)\circ\Delta_k(\theta)\circ\sigma_k(\theta)$. Furthermore, we have the decomposition:

$$\frac{N_k+1}{\mathrm{vol}_\omega(X)}\Delta_{\diamond_k} = \oplus_{\theta\in\mathrm{Rep}(G)}\mathrm{id}_{R_\theta}\otimes\Delta_{\diamond_k}(\theta)\ ,$$

where $\hat{\Delta}_{\diamond_k}(\theta)=\beta_k(\theta)\circ\Delta_k(\theta)$. The relation $T_k=Q_k\circ\beta_k$ gives $T_k(\theta)=Q_k(\theta)\circ\beta_k(\theta)$. Of course, the coherent states in this case can be determined explicitly through Perelomov's method, and the operators $Q_k(\theta),T_k(\theta),\beta_k(\theta)$ etc. can be expressed in terms of the representation theory of G.

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[15]This recovers the observation of [31] that Δ preserves Σ_k on all flag manifolds when using the classical Berezin quantization with respect to their homogeneous Kähler metric.

Remark. A particularly simple case arises when all non-zero multiplicities $m_k(\theta)$ in the decomposition (6.4) equal one. Then all non-vanishing spaces $W_k(\theta)$ are one-dimensional and can be identified with the space $\mathbb{C}$ of complex numbers upon fixing a normalized vector in each of them. The non-vanishing components $Q_k(\theta), T_k(\theta), \sigma_k(\theta), \beta_k(\theta)$ are simply complex numbers, and we find:

$$\Delta_{\diamond_k} = \frac{\mathrm{vol}_\omega(X)}{N_k + 1} \oplus_{\theta \in \mathrm{Rep}(G):m_k(\theta)\neq 0} \beta_k(\theta)\Delta_k(\theta)\mathrm{id}_{R_\theta}$$

and:

$$\hat{\Delta}_k = \frac{\mathrm{vol}_\omega(X)}{N_k + 1} \oplus_{\theta \in \mathrm{Rep}(G):m_k(\theta)\neq 0} \beta_k(\theta)\Delta_k^B(\theta)\mathrm{id}_{R_\theta}$$

since in this case we have $\hat{\Delta}_k(\theta) = \beta_k(\theta)\Delta_k^B(\theta)$. If one furthermore has $\beta_k(\theta) \in \mathbb{R}_+$ for all k and θ, then it follows that both Δ_k and $\Delta_{\diamond_k}$ are $\langle\,,\,\rangle_B$-Hermitian and positive and both $\hat{\Delta}_k$ and Δ_k^B are $\langle\,,\,\rangle_{\mathrm{HS}}$-Hermitian and positive. In such a situation, one can use the Berezin push of Δ as a fuzzy Laplacian since it is self-adjoint with respect to the Hilbert-Schmidt scalar product. As we shall see in the next subsection, this very particular case arises e.g. for $X = \mathbb{P}^n$, when the Berezin push Δ_k^B coincides with the second Casimir operator of $G = \mathrm{U}(n+1)$ in the representation $\mathrm{End}(E_k)$.

6.3 The quantum laplacian on $\mathbb{P}^n$

As shown in section 4, classical Berezin quantization of $(\mathbb{P}^n, \omega_{\mathrm{FS}})$ with quantum line bundle H agrees with its Berezin-Bergman quantization and with the construction of fuzzy projective spaces used in the fuzzy geometry literature [17, 29]. Since $(\mathbb{P}^n, \omega_{\mathrm{FS}})$ is the Kähler homogeneous space $\mathrm{U}(n+1)/(\mathrm{U}(n) \times \mathrm{U}(1))$, and since the hyperplane bundle is equivariant, the spaces $E_k = H^0(H^k)$ carry a unitary representation ρ_k of $\mathrm{U}(n+1)$. Thinking of E_k as the space R_k of homogeneous polynomials of degree k in $n+1$ variables, it is clear that ρ_k is the totally symmetric irreducible representation, which has Dynkin labels $(k, 0, \ldots, 0)$. The space $\mathrm{End}(E_k)$ thus forms the (reducible) tensor product representation $(k, 0, \ldots, 0) \otimes (0, \ldots, 0, k)$, which decomposes into irreducibles as:

$$\mathrm{End}(E_k) \cong \bigoplus_{\ell=0}^k (\ell, 0, \ldots, 0, \ell) \ \ .$$

Notice that all subspaces in this decomposition appear with multiplicity one.

The usual fuzzy Laplacian is given by the second Casimir $\hat{C}_2^{(k)}$ of $\mathrm{U}(n+1)$ in the representation $\hat{\rho}_k = \rho_k^* \otimes_\mathbb{C} \rho_k$ on $\mathrm{End}(E_k) = E_k^* \otimes_\mathbb{C} E_k$. The explicit form of this operator in terms of annihilation and creation operators follows from the Schwinger construction (cf. [30]):

$$\hat{C}_2^{(k)}(C) = \sum_a [\hat{\mathcal{L}}^a, [\hat{\mathcal{L}}^a, C]] \ , \quad \hat{\mathcal{L}}^a = \sum_{i,j} \hat{a}_i^\dagger \frac{\tau_{ij}^a}{2} \hat{a}_j \ \ . \tag{6.11}$$

Here τ_{ij}^a are the Gell-Mann matrices of $su(n+1)$ with normalization fixed by the Fierz identity:

$$\sum_a \tau_{ij}^a \tau_{kl}^a = 2\left(\delta_{il}\delta_{jk} - \frac{1}{n+1}\delta_{ij}\delta_{kl}\right) \ \ .$$

Let us first show that $\hat{C}_2^{(k)}$ agrees with Δ_k^B.

Lemma. Let $P_x^{(k)}$ be the coherent projectors of $\mathbb{P}^n$ at level k and consider the vector valued function $P^{(k)} : X \to \operatorname{End}(E_k)$, $P^{(k)}(x) := P_x^{(k)}$. Then $\hat{C}_2^{(k)} \circ P^{(k)} = \Delta(P^{(k)})$, i.e.

$$\hat{C}_2^{(k)}(P_x^{(k)}) = \Delta P_x^{(k)} \quad (x \in X) \quad .$$

Proof. Direct computation gives:

$$\hat{C}_2^{(k)}(C) = \hat{N}(\hat{N} + n)C - \sum_{ij} \hat{a}_j^\dagger \hat{a}_i C \hat{a}_i^\dagger \hat{a}_j \quad ,$$

where $\hat{N} := \sum_i \hat{a}_i^\dagger \hat{a}_i$ is the number operator. For simplicity, let us now restrict to the case of $\mathbb{P}^1$ (the proof for $n > 1$ follows along the same lines). With the homogeneous coordinates denoted by z_0, z_1, the Laplace operator on the patch[16] $z_1 \neq 0$ with local coordinate $z := z_0/z_1$ reads as:

$$\Delta f := (1 + z\bar{z})^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} f \quad .$$

Introducing the quantities:

$$F_{rs}^k = \frac{z^r \bar{z}^s}{(1 + z\bar{z})^k} \quad \text{and} \quad M_{rs}^k = \frac{1}{k!} (\hat{a}_0^\dagger)^r (\hat{a}_1^\dagger)^{k-r} |0\rangle \langle 0| (a_0^\dagger)^s (a_1^\dagger)^{k-s} \quad , \tag{6.12}$$

we have:

$$P_x^{(k)} = \sum_{r,s=0}^k \binom{k}{r} \binom{k}{s} F_{rs}^k M_{rs}^k \quad .$$

One easily checks the identities:

$$\Delta F_{rs}^k = k(k+1) F_{rs}^k - rs F_{r-1,s-1}^k - (k-r)(k-s) F_{rs}^k - rs F_{rs}^k - (k-r)(k-s) F_{r+1,s+1}^k$$

and:

$$\hat{C}_2^{(k)}(M_{rs}^k) = k(k+1) M_{rs}^k - rs M_{r-1,s-1}^k - (k-r)(k-s) M_{rs}^k - rs M_{rs}^k - (k-r)(k-s) M_{r+1,s+1}^k \quad .$$

It is also easy to check that:

$$\sum_{r,s=0}^k \binom{k}{r} \binom{k}{s} F_{rs}^k rs M_{r-1,s-1}^k = \sum_{r,s=0}^{k-1} \binom{k}{r+1} \binom{k}{s+1} F_{r+1,s+1}^k (r+1)(s+1) M_{rs}^k$$

$$= \sum_{r,s=0}^k \binom{k}{r} \binom{k}{s} F_{r+1,s+1}^k (k-r)(k-s) M_{rs}^k \quad .$$

The same identity holds when F_{rs}^k and M_{rs}^k are interchanged. Putting everything together, one finds $\hat{C}_2^{(k)}\left(P_x^{(k)}\right) = \Delta P_x^{(k)}$.

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[16]Proving the identity on a single patch is evidently sufficient, as we are missing only one point on $\mathbb{P}^1$ and all the functions involved are in $\mathcal{C}^\infty(\mathbb{P}^1)$.

Proposition. For every $k \geq 1$, the Berezin push Δ_k^B of the truncated Laplacian Δ_k of $\mathbb{P}^n$ coincides with the second Casimir of $\mathrm{U}(n+1)$ in the representation $\mathrm{End}(E_k)$

$$\Delta_k^B = \hat{C}_2^{(k)} \quad .$$

Proof. Using the Lemma, we compute:

$$\sigma\left(\Delta_k^B(C)\right)(x) = \Delta_k \,\mathrm{tr}\left(P_x^{(k)} C\right) = \mathrm{tr}\left(\hat{C}_2^{(k)}(P_x^{(k)}) C\right)$$
$$= \mathrm{tr}\left(P_x^{(k)} \hat{C}_2^{(k)}(C)\right) = \sigma\left(\hat{C}_2^{(k)}(C)\right)(x) \quad .$$

The next to last equality holds due to the form $\hat{C}_2^{(k)}(C) = \sum_a [\hat{\mathcal{L}}^a, [\hat{\mathcal{L}}^a, C]]$ of the Casimir in the representation $\mathrm{End}(E_k)$. The conclusion of the proposition now follows by using injectivity of σ.

We next consider the Berezin-Toeplitz lift of Δ. Since $\mathbb{P}^n$ is a Kähler homogeneous space, the difference between the Berezin-Toeplitz lift and the Berezin push of Δ_k is an ℓ-dependent rescaling on the eigenfunctions[17] $Y_{\ell M}$. Let us show this more explicitly. To be concise, we will rely on results presented e.g. in [17, 30], to which we refer the reader for further details.

First, note that $\mathrm{End}(E_k)$ is spanned by operators $\hat{Y}_{\ell M}$, $\ell = 0, \ldots, k$, called *polarization tensors*. These are the operator analogues of hyperspherical harmonics and satisfy:

$$\hat{C}_2^{(k)}(\hat{Y}_{\ell M}) = \ell(\ell + n) \quad . \tag{6.13}$$

The multi-index M captures the same indices as for $Y_{\ell M}$. The polarization tensors are orthogonal with respect to the Hilbert-Schmidt scalar product, and we choose the normalization (cf. [30]):

$$\frac{1}{\dim(\mathrm{End}(E_k))} \mathrm{tr}(\hat{Y}_{\ell M} \hat{Y}_{\ell' M'}) = \delta_{\ell \ell'} \delta_{MM'} \quad .$$

Direct computation gives the relation [30]:

$$P_x^{(k)} = \sum_{k, \ell, M} T_{k,n}^{1/2}(\ell) Y_{\ell M}(x) \hat{Y}_{\ell M} \quad , \quad T_{k,n}(\ell) := \frac{k!(k+n)!}{n!(k-\ell)!(k+\ell+n)!} \quad , \tag{6.14}$$

from which we conclude that the Berezin symbols of the polarization tensors are:

$$\sigma(\hat{Y}_{\ell M}) = \mathrm{tr}(P_x^{(k)} \hat{Y}_{\ell M}) = \dim(\mathrm{End}(E_k)) T_{k,n}^{1/2}(\ell) Y_{\ell M} \quad .$$

Using this, one readily computes:

$$\frac{1}{\dim(\mathrm{End}(E_k))} \mathrm{tr}(\hat{Y}_{\ell' M'} T(Y_{\ell M})) = \mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^n) T_{k,n}^{1/2}(\ell) \delta_{\ell \ell'} \delta_{MM'} \quad ,$$

---

[17]For brevity, we will always denote the hyperspherical harmonics on $\mathbb{P}^n$ by $Y_{\ell M}$, where ℓ is the angular momentum $\Delta Y_{\ell M} = \ell(\ell + n) Y_{\ell M}$ and M is a multi-index capturing all further labels. We work with the normalization $\frac{1}{\mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^n)} \int \frac{\omega_{\mathrm{FS}}^n}{n!} Y_{\ell M} Y_{\ell' M'} = \delta_{\ell \ell'} \delta_{MM'}$. A detailed discussion can be found e.g. in [30].

from which it follows that the Toeplitz quantization of $Y_{\ell M}$ is given by:

$$T(Y_{\ell M}) = \mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^n) T_{k,n}^{1/2}(\ell) \hat{Y}_{\ell M} \ .$$

Hence the numbers $\beta_k(\theta_\ell)$ (cf. section 6.2), where θ_ℓ refers to the irrep of $\mathrm{SU}(n+1)$ with Dynkin labels $(\ell, 0, \ldots, 0, \ell)$, are given by:

$$\beta_k(\theta_\ell) = \frac{\sigma(T(Y_{\ell M}))}{Y_{\ell M}} = \dim(\mathrm{End}(E_k)) \mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^n) T_{k,n}(\ell) \ .$$

Note that $\beta_k(\theta_\ell)$ are real and positive. As explained in the previous subsection, this means that both the Berezin pull and the Berezin-Toeplitz lift are reasonable candidates for the quantized Laplacian in this case. It is now trivial to compute:

$$\mathrm{tr}(\hat{Y}_{\ell M} \hat{C}_2^{(k)} \hat{Y}_{\ell' M'}) = \dim(\mathrm{End}(E_k)) \ell(\ell+n) \delta_{\ell\ell'} \delta_{MM'}$$

and:

$$\mathrm{tr}(\hat{Y}_{\ell M} \hat{\Delta}_k \hat{Y}_{\ell' M'}) = \dim(\mathrm{End}(E_k))^2 \mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^n) T_{k,n}(\ell) \ell(\ell+n) \delta_{\ell\ell'} \delta_{MM'} \ .$$

It follows that on $\mathbb{P}^n$, $\hat{\Delta}_k$ and $\Delta_k^B = \hat{C}_2^{(k)}$ are related to each other via the positive ℓ-dependent rescaling factors:

$$\tilde{T}_{k,n}(\ell) := \dim(\mathrm{End}(E_k)) \mathrm{vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^n) T_{k,n}(\ell) \ . \tag{6.15}$$

6.4 Approximating the spectrum of the laplacian on Fermat curves

In spite of its shortcomings, the truncated Laplacian (equivalently, its Berezin push) can be used to approximate the spectrum of the classical Laplacian. When the generalized Berezin quantization is chosen such that $\overline{\cup_{k=0}^{\infty} \Sigma_k} = \mathcal{C}^\infty(X)$, the operators Δ_k can be viewed as approximations to the full Laplacian and the spectrum of the latter can be approximated by computing the spectra of Δ_k. This happens, for example, when Σ_k are the symbol spaces of the Berezin-Bergman quantization, since in that case the union of Σ_k is dense in $\mathcal{C}^\infty(X)$ (see section 5).

To be more explicit, we will consider the Berezin-Bergmann quantization of *Fermat curves*, i.e. the projective algebraic curves $X_p \subset \mathbb{P}^2$ given by the equation:

$$f(z_0, z_1, z_2) \ := \ z_0^p + z_1^p + z_2^p \ = \ 0 \ , \tag{6.16}$$

where (z_0, z_1, z_2) are homogeneous coordinates on $\mathbb{P}^2$. These curves are non-singular and of genus $(p-1)(p-2)/2$. In the following, we will restrict our attention to the cases $p = 2$ (the *conic*) and $p = 3$ (the Fermat elliptic curve).

We endow X_p with the Bergman metric given by the pull-back of the Fubini-Study metric via the inclusion map $i : X_p \hookrightarrow \mathbb{P}^2$. It will be convenient to cover[18] $\mathbb{P}^2$ by the patches U_{ijk}:

$$U_{ijk} \ := \ \{z \in \mathbb{P}^2 \mid |z_i| \geq |z_j| \geq |z_k|\} \ , \tag{6.17}$$

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[18]This covering together with the integration method we use has been considered, e.g., in [32].

where we choose the normalization $|z_i| = 1$ and denote the resulting coordinates by $z_m^{(ijk)}$. One easily checks that the patches intersect only on their boundaries.

Note that $x = z_k^{(ijk)}$ is a good local coordinate on the patch U_{ijk}. The pull-back of the Fubini-Study metric is easily calculated by noting that:

$$\frac{\partial z_j^{(ijk)}}{\partial x} = -\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial z_j^{(ijk)}}\right)^{-1} \ ,$$

which is a consequence of $f = 0$. The patches are chosen such that the pull-back $i^*\omega(x) =: w(x)dx \wedge d\bar{x}$ is always well defined. The Laplacian is given by:

$$\Delta \ := \ \frac{1}{w(x)} \frac{\partial}{\partial x} \frac{\partial}{\partial \bar{x}} \ .$$

We approximate the integrals in (6.3) by summing over the integrand evaluated at a random sample of N points on each patch and summing over patches. To generate the sample points, we proceed as follows, cf. [32]. On the patch U_{ijk} we pick a point $z_k^{(ijk)}$ in the unit disk: $|z_k^{(ijk)}| < 1$. The coordinate $z_j^{(ijk)}$ is evaluated as:

$$z_j^{(ijk)} \ = \ \phi_{\text{rnd}} \left[-1 - (z_k^{(ijk)})^3\right]^{\frac{1}{p}} \ ,$$

where ϕ_{rnd} is a uniformly chosen random p-th root of unity. If $1 \geq |z_j^{(ijk)}| \geq |z_k^{(ijk)}|$, we include the point $(1, z_j^{(ijk)}, z_k^{(ijk)})$ in the set of sample points; otherwise we pick a new one. We then use the formula:

$$\int_{U_{ijk}} \omega f(x) \approx \frac{1}{6N} \sum_{n=1}^{N} f(x_n) w(x_n) \,,$$

The total integral is obtained by summing over all 6 patches.

Our set Σ_k is the set of symbols of Berezin-Bergman quantization at level k. Thus we consider the polynomial ring $B = \oplus_{k=0}^{\infty} B_k$ on $\mathbb{P}^2$ and the vanishing ideal $I = (f) = \oplus_{k=0}^{\infty} I_k$ with $I_k \subset B_k$. The space of endomorphisms of $E_k = B_k/I_k$ is identified with Σ_k. A basis (e_i) for Σ_k is constructed from a set of monomials (χ_α) of degree k forming a basis of E_k by considering all pairs $\nu_k \chi_\alpha \bar{\chi}_\beta$, where ν_k is the normalization factor:

$$\nu_k \ := \ \frac{1}{||z||^{2k}} \ = \ \frac{1}{\left(1 + z_j^{(ijk)}\bar{z}_j^{(ijk)} + z_k^{(ijk)}\bar{z}_k^{(ijk)}\right)^k} \ . \tag{6.18}$$

The projector $\pi_k : \mathcal{C}^\infty(X_p) \to \Sigma_k$ is defined by the integral expression (6.3).

Given an arbitrary, not necessarily orthonormal, basis (e_i) of Σ_k, we expand the eigenvalue equation $\Delta f_m = \lambda_m f_m$ in the following manner:

$$\sum_i \langle e_j|\Delta|e_i\rangle\langle e_i|f_m'\rangle = \lambda_m \sum_i \langle e_j|e_i\rangle\langle e_i|f_m'\rangle \ , \tag{6.19}$$
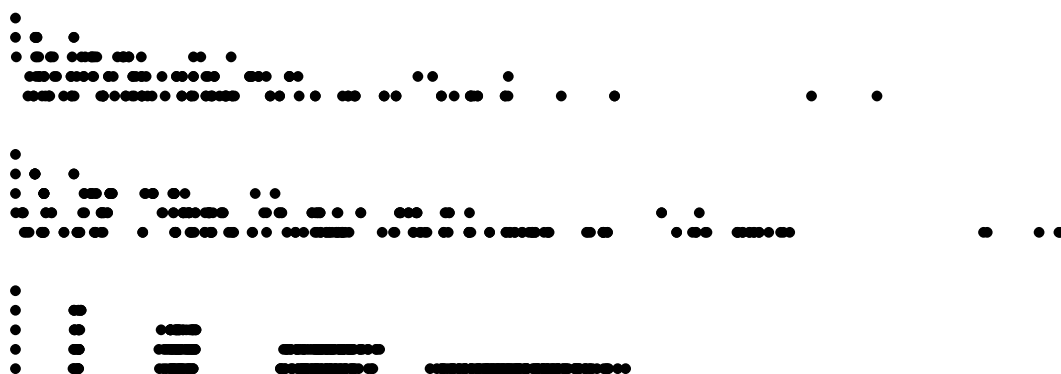
Figure 1: Spectra of the Laplace operator on the Fermat curves X_2, X_3 and $\mathbb{P}^2$ (from top to bottom) for $k = 0, 1, 2, 3, 4$. The spreading of the eigenvalues is related to numerical errors.

where $\langle\,,\,\rangle$ denotes again the scalar product with respect to the L^2-norm and f'_m is the (unique) function such that $f_i = \sum_i e_i \langle e_i | f'_m \rangle$. After defining the vector $\vec{f'}_m = (\langle e_i | f'_m \rangle_i)$, the eigenvalue problem (6.19) turns into the form:

$$A \vec{f'}_m = \lambda_m B \vec{f'}_m \ ,$$

where A and B are matrices, B being invertible. The eigenvalues λ_m of the Laplace operator are therefore the eigenvalues of the matrix $B^{-1}A$. Note that this procedure is in fact equivalent to the one used in [33].

The numerical results[19] for $k \leq 2$ are presented in tables 1 and 2. For comparison, we ran our algorithm also for the space $\mathbb{P}^2$ itself, choosing the same patches (in that case, E_k is the space of linear endomorphisms of B_k); the results of this are shown in table 3. In all cases, the integration was performed using $10,000$ points per patch. The spectra of the Laplace operators for $k \leq 4$ are displayed in figure 1.

6.5 Fuzzy real scalar field theory on compact Hodge manifolds

Ordinary real scalar field theory on (X, ω) is defined by the Euclidean action functional:

$$S[\phi] \ = \ \frac{1}{\text{vol}_\omega(X)} \int_X \frac{\omega^n}{n!} \left(\phi \Delta \phi + V(\phi) \right) \ \ (\phi \in \mathcal{C}^\infty(X, \mathbb{R})) \ , \tag{6.20}$$

where Δ is the Laplace operator of (X, ω), and $V(\phi) = \sum_{s=0}^d a_s \phi^s$ is a polynomial in ϕ of degree d with real coefficients $a_k \in \mathbb{R}$. Notice that we include a possible mass term for ϕ as a quadratic contribution to V. Since X is a compact space, potentials V of odd degree are in principle allowed, though the consistency of the corresponding quantum theory depends on a detailed analysis of quantum effects.

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[19]Our computations are merely a demonstration of principle, as the algorithm is run on a laptop using Mathematica. Switching to C and using more powerful computers, one can easily increase the precision.

k	Eigenvalues of Δ						
0	0						
1	3.00086	3.00081					
	1.00611	1.00588	1.00404	1.00404	0.992317	0.992314	
	0						
2	11.0669	9.51623	9.13809				
	6.46006	5.8219	5.8219	5.5911	5.50849	5.50849	5.24445
	4.17542	4.06504	3.9937	3.90068	3.90068		
	3.58166	3.3962	2.91584				
	1.95561	1.83634	1.79089	1.19609	1.10872	1.0353	
	0.0465755						

Table 1: Results for the eigenvalues of the Laplace operator on X_2.

k	Eigenvalues of Δ						
0	0						
1	3.00086	3.00081					
	1.00611	1.00588	1.00404	1.00404	0.992317	0.992314	
	0						
2	13.3283	12.3062					
	8.70644	8.1688	8.16022	8.14907	8.13025	8.12288	8.06782
	7.08575	7.05761	7.04119	7.04118	6.67228	6.63273	
	4.97054	4.9572	4.85558	4.84641	4.84596	4.84476	4.16108
	4.10807	3.92685	3.90904	3.87988	3.85214	3.53332	3.52194
	1.47074	1.46942	1.46242	1.4616	1.45804	1.45203	
	0.00002						

Table 2: Results for the eigenvalues of the Laplace operator on X_3.

k	Eigenvalues of Δ							
0	0							
1	3.37386	3.35758	3.34758	3.23647	3.23203	3.03376	3.00594	2.99832
	0							
2	9.26534	9.26534	9.231	9.16842	9.08967	9.08967	9.08707	9.03964
	8.95108	8.77926	8.60043	8.59167	8.43174	8.40971	8.37658	8.3692
	8.30439	8.28543	8.25084	8.25084	8.1738	7.97861	7.95145	7.95145
	7.94024	7.94024	7.48435					
	3.30249	3.26888	3.25238	3.23301	3.22454	3.06392	3.02943	3.01363
	0							

Table 3: Eigenvalues of the Laplace operator on $\mathbb{P}^2$ as computed by our algorithm. The exact eigenvalues joining the spectrum at level k are given by $k(k+2)$ with a degeneracy of $(1+k)^3$.

The discussion in the previous subsections allows us to define a "fuzzy" version of the action (6.20) as follows:

$$S_k(\Phi) := \text{tr}\left[\Phi\hat{\Delta}_k(\Phi) + V(\Phi)\right] \ , \tag{6.21}$$

where $\Phi \in \text{End}(E_k)$. The reality condition $\bar{\phi} = \phi$ is replaced by the $\langle , \rangle_{\text{HS}}$-hermiticity requirement $\Phi^\dagger = \Phi$. Using the relation $\sigma_k(\hat{\Delta}_k\Phi) = \Delta_{\diamond_k}\sigma_k(\Phi)$ (see (3.37)), we find:

$$S_k(\Phi) = S_k^\diamond(\sigma_k(\Phi)) \ , \tag{6.22}$$

where:

$$S_k^\diamond[\phi] := \frac{1}{\text{vol}_\omega(X)}\int_k [\phi \diamond_k \Delta_{\diamond_k}\phi + V_{\diamond_k}(\phi)] = \frac{1}{\text{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}\epsilon_k [\phi \diamond_k \Delta_{\diamond_k}\phi + V_{\diamond_k}(\phi)]$$

for $\phi \in \Sigma_k$ such that $\bar{\phi} = \phi$. Here $V_{\diamond_k}(\phi) := \sum_{s=0}^{2d} a_s \phi^{\diamond_k s}$, where $\phi^{\diamond_k s} := \phi \diamond_k \ldots \diamond_k \phi$ (s times) and we used relation (3.28).

Working with the finite dimensional space $\Sigma_k \cong \text{End}(E_k)$ reduces the functional integral $\int \mathcal{D}[\phi]$ in the definition of the partition function:

$$Z = \int \mathcal{D}[\phi] \ e^{-S[\phi]} \tag{6.23}$$

to a well-defined finite dimensional integral Z_k. (On $\mathbb{P}^n$, for example, the functional measure $\mathcal{D}[\Phi]$ becomes the Dyson measure on the space of Hermitian operators on E_k). Hence Z_k provide *regularizations* of the quantum field theory defined by (6.20). These regularized field theories are known in the literature as fuzzy scalar field theories.[20]

Remark. Let ρ_k be the Toeplitz quantization of the function $\frac{1}{\text{vol}_\omega(X)\epsilon_k}$ at level k:

$$\rho_k := T_k\left(\frac{1}{\text{vol}_\omega(X)\epsilon_k}\right) = \frac{1}{\text{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}P_x^{(k)} \in \text{End}(E_k) \ . \tag{6.24}$$

Clearly ρ_k is Hermitian and strictly positive on $(E_k, \langle , \rangle_k)$. Furthermore $\text{tr}(\rho_k) = 1$, so ρ_k is a density operator on E_k. For any operator $C \in \text{End}(E_k)$, we have:

$$\text{tr}(\rho_k C) = \frac{1}{\text{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}\sigma_k(C) \ .$$

Hence the operator ρ_k allows us to remove the epsilon function from the integral.

7. Directions for further research

Generalized Berezin quantization raises a series of natural questions about the asymptotic behavior of the quantization maps Q_k for large k as a function of the defining sequence of scalar products $(,)_k$. In particular, one would like to know what conditions should be imposed on the large k behavior of these scalar products in order to ensure that the generalized quantization prescription induces a formal star product on $\mathcal{C}^\infty(X)$ and thus defines a

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[20]See [29] for more details on this point.

formal deformation quantization. Other natural questions involve the relation with Chow-Mumford stability and K-stability and with approximation theorems for Kähler metrics of constant scalar curvature. An important set of applications concerns the quantization of toric varieties, and the extension to the singular case.

The definition of the fuzzy Laplace operator as the Berezin-Toeplitz lift of the classical Laplacian remains somewhat ad hoc. A better understanding of the quantization of the classical Laplacian Δ seems to require the quantization of differential forms and the construction of a quantum analogue of a volume form.

It would also be interesting to examine the relevance of our general quantized spaces within string theory. In particular, one could study the extension of the Myers effect [34] to more general Hodge manifolds.

Acknowledgments

8. Glossary of notation

Notation is in page order of definition (rather than first appearance) excluding the introduction. Note that $\langle \, , \, \rangle_k$ indicates an *induced* Hermitian scalar product on E_k depending on the situation (usually the L^2-scalar products $\langle \, , \, \rangle_k^h$) while $(\, , \,)_k$ is an arbitrary sequence of Hermitian scalar products on E_k used in the generalized quantization procedures.

Notation	Explanation	Page (Eqn)
X	compact complex manifold, usually Kähler and/or Hodge	5
(X, L)	polarized complex manifold	7
$R(X, L)$	homogeneous coordinate ring of (X, L) embedded in $\mathbb{P}V$	10
ω	Kähler form, usually L-polarized	5
ω_{FS}	Kähler form of Fubini-Study metric	11, 41
(X, L, ω)	polarized Hodge manifold	5
L^k	$L^k := L^{\otimes k}$	5
$\Gamma(L^k)$	space of smooth sections of L^k (contains E_k)	6
(L, h)	Hermitian holomorphic line bundle on (X, ω), "prequantum bundle"	6
(X, ω, L, h)	prequantized Hodge manifold	6
$\mathrm{Aut}(X, \omega, L, h)$	automorphism group of a prequantized Hodge manifold	6
$\mathrm{Aut}_{L,h}(X, \omega, L, h)$	$\mathrm{Aut}(X, \omega, L, h)/\, \mathrm{U}(1)$, subgroup of isometries admitting a lift	7
∇	Chern connection associated to (L, h)	5
∇_k	Chern connection associated to (L^k, h_k)	6
F	curvature of ∇	5
F_k	curvature of ∇_k	6
h_k	$h_k := h^{\otimes k}$, Hermitian scalar product on L^k	6
h_B	Bergman metric	8 (2.7)
h_{FS}	Hermitian metric on H, $h_{\mathrm{FS}}^k := h_{\mathrm{FS}}^{\otimes k}$	41 (4.16)

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